NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

COURSE MATERIALS

MA 201-LINEAR ALGEBRA AND COMPLEX ANALYSIS

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Computer Science and Engineering

  M.Tech in Computer Science and Engineering

  M.Tech in Cyber Security
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Producing Highly Competent, Innovative and Ethical Computer Science and Engineering Professionals to facilitate continuous technological advancement.

DEPARTMENT MISSION

1. To Impart Quality Education by creative Teaching Learning Process
2. To Promote cutting-edge Research and Development Process to solve real world problems with emerging technologies.
3. To Inculcate Entrepreneurship Skills among Students.
4. To cultivate Moral and Ethical Values in their Profession.
5.

PROGRAMME EDUCATIONAL OBJECTIVES

PEO1: Graduates will be able to Work and Contribute in the domains of Computer Science and Engineering through lifelong learning.
PEO2: Graduates will be able to Analyse, design and development of novel Software Packages, Web Services, System Tools and Components as per needs and specifications.
PEO3: Graduates will be able to demonstrate their ability to adapt to a rapidly changing environment by learning and applying new technologies.
PEO4: Graduates will be able to adopt ethical attitudes, exhibit effective communication skills, Teamworkand leadership qualities.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering

problems.

2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1: Ability to Formulate and Simulate Innovative Ideas to provide software solutions for Real-time Problems and to investigate for its future scope.

PSO2: Ability to learn and apply various methodologies for facilitating development of high quality System Software Tools and Efficient Web Design Models with a focus on performance optimization.

PSO3: Ability to inculcate the Knowledge for developing Codes and integrating hardware/software products in the domains of Big Data Analytics, Web Applications and Mobile Apps to create innovative career path and for the socially relevant issues.

COURSE OUTCOME

After the completion of the course students will be able to:

CO 1: identify analytic functions and Harmonic functions.
CO 2: identify conformal mappings and find regions that are mapped under certain transformation.
CO 3: find integral of complex functions.
CO 4: evaluate real definite Integrals as application of Residue Theorem.
CO 5: solve any given system of linear equations.
CO 6: find the Eigen values of a matrix and how to diagonalize a matrix.

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PS01	PSO2	PSO3
CO 1	3	3	3	3	2	1	-	-	1	2	-	2	1	1	-
CO 2	3	3	3	3	2	1	-	-	1	2	-	2	-	-	1
CO 3	3	3	3	3	2	1	-	-	-	2	-	2	1	-	1
CO 4	3	3	3	3	2	1	-	-	-	2	-	2	1	1	1
CO 5	3	3	3	3	2	1	-	-	-	2	-	2	-	1	-
CO 6	3	3	3	3	2	1	-	-	-	2	-	2	1	-	1

Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1

SYLLABUS

Course No.	Course Name	L-T-P - Credits	Year of Introduction
MA201	LINEAR ALGEBRA AND COMPLEX ANALYSIS	3-1-0-4	2016

Prerequisite : Nil

Course Objectives

COURSE OBJECTIVES

- To equip the students with methods of solving a general system of linear equations.
- To familiarize them with the concept of Eigen values and diagonalization of a matrix which have many applications in Engineering.
- To understand the basic theory of functions of a complex variable and conformal Transformations.

Syllabus

Analyticity of complex functions-Complex differentiation-Conformal mappings-Complex integration-System of linear equations-Eigen value problem

Expected outcome .

At the end of the course students will be able to

(i) solve any given system of linear equations

(ii) find the Eigen values of a matrix and how to diagonalize a matrix

(iii) identify analytic functions and Harmonic functions.

(iv)evaluate real definite Integrals as application of Residue Theorem

(v) identify conformal mappings(vi) find regions that are mapped under certain Transformations

Text Book:

Erwin Kreyszig: Advanced Engineering Mathematics, 10th ed. Wiley

References:

1.Dennis g Zill&Patric D Shanahan-A first Course in Complex Analysis with Applications-Jones&Bartlet Publishers

2.B. S. Grewal. Higher Engineering Mathematics, Khanna Publishers, New Delhi.

3.Lipschutz, Linear Algebra,3e **(** Schaums **Series)**McGraw Hill Education India 2005

4.Complex variables introduction and applications-second edition-Mark.J.Owitz-Cambridge Publication

Course Plan

Module	Contents	Hours	Sem. Exam Marks
I	<u>Complex differentiation Text 1[13.3,13.4]</u> Limit, continuity and derivative of complex functions	3	
	Analytic Functions	2	
	Cauchy–Riemann Equation(Proof of sufficient condition of analyticity & C R Equations in polar form not required)-Laplace's Equation	2	
	Harmonic functions, Harmonic Conjugate	2	15%
II	<u>Conformal mapping: Text 1[17.1-17.4]</u> Geometry of Analytic functions Conformal Mapping,	1	
	Mapping $w = z^2$ conformality of $w = e^z$.	2	15%

	The mapping $w = z + \frac{1}{z}$ Properties of $w = \frac{1}{z}$ Circles and straight lines, extended complex plane, fixed points Special linear fractional Transformations, Cross Ratio, Cross Ratio property-Mapping of disks and half planes Conformal mapping by $w = \sin z$ & $w = \cos z$ (Assignment: Application of analytic functions in Engineering)	1 3 3	
FIRST INTERNAL EXAMINATION			
III	Complex Integration. Text 1[14.1-14.4] [15.4&16.1] Definition Complex Line Integrals, First Evaluation Method, Second Evaluation Method Cauchy's Integral Theorem(without proof), Independence of path(without proof), Cauchy's Integral Theorem for Multiply Connected Domains (without proof) Cauchy's Integral Formula- Derivatives of Analytic Functions(without proof)Application of derivative of Analytical Functions Taylor and Maclaurin series(without proof), Power series as Taylor series, Practical methods(without proof) Laurent's series (without proof)	2 2 2 2 2	15%
IV	Residue Integration Text 1 [16.2-16.4] Singularities, Zeros, Poles, Essential singularity, Zeros of analytic functions Residue Integration Method, Formulas for Residues, Several singularities inside the contour Residue Theorem. Evaluation of Real Integrals (i) Integrals of rational functions of $\sin\theta$ and $\cos\theta$ (ii)Integrals of the type $\int_{-\infty}^{\infty} f(x)dx$ (Type I, Integrals from 0 to ∞) (Assignment : Application of Complex integration in Engineering)	2 4 3	15%
SECOND INTERNAL EXAMINATION			
V	Linear system of Equations Text 1(7.3-7.5) Linear systems of Equations, Coefficient Matrix, Augmented Matrix Gauss Elimination and back substitution, Elementary row operations, Row equivalent systems, Gauss elimination-Three possible cases, Row Echelon form and Information from it.	1 5	20%

	Linear independence-rank of a matrix Vector Space-Dimension-basis-vector space**R**^{3} Solution of linear systems, Fundamental theorem of non-homogeneous linear systems(Without proof)-Homogeneous linear systems (Theory only	2 1	
VI	Matrix Eigen value Problem Text 1.(8.1,8.3 &8.4) Determination of Eigen values and Eigen vectors-Eigen space Symmetric, Skew Symmetric and Orthogonal matrices –simple properties (without proof) Basis of Eigen vectors- Similar matrices Diagonalization of a matrix-Quadratic forms- Principal axis theorem(without proof) (Assignment-Some applications of Eigen values(8.2))	3 2 4	20%
END SEMESTER EXAM			

QUESTION BANK

MODULE I				
Q:NO:	**QUESTIONS**	**CO**	**KL**	**PAGE NO:**
1	Show that the function $u = e^{-2xy}\sin(x^2 - y^2)$ is harmonic. Find the conjugate function V and express $u + iv$ as an analytic function of z.	CO1	K2	4
2	Show that $u=x^2-y^2-y$ is harmonic. Also find the corresponding conjugate harmonic function.	CO1	K5	10
3	If $V= e^x (x \sin y + y \cos y)$, find an analytic function f(z)=u+iv	CO1	K2	12
4	Determine the analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$	CO1	K3	15

5	Prove that the function $f(z)=\begin{cases}\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2} & z\neq 0\\ 0 & z=0\end{cases}$ is not analytic at z=0 although C-R equations are satisfied at that point.	CO 1	K3	16
6	Prove that $v=3x^2y+x^2-y^3-y^2$ is harmonic. Also find the harmonic conjugate	CO 1	K2	18
7	Show that $\sqrt{\lvert xy\rvert}$ is not regular at origin even though C-R equations are satisfied	CO 1	K5	20
8	Prove that the function $e^x(x\cos y - y\sin y)$ is harmonic hence find its harmonic conjugate	CO 1	K4	18
9	Find the analytic function $f(z)\ =u+iv$ where $u-v=(x-y)(x^2+4\text{xy}+y^2)$	CO 1	K2	26
10	If $f(z)\ =u+iv$ is analytic, Prove that u=constant and v=constant are families of curves cutting orthogonally.	CO 1	K5	14
	MODULE II			
1	Find the image of the regions 2<|z|<3 and |argz|<$\frac{\pi}{4}$ under the transformation $w=z^2$ and plot it	CO 2	K5	38
2	Find the image of the following infinite strips under the mapping $w=\frac{1}{z}$	CO 2	K2	41
3	Find the image of the region **|z-$\frac{1}{3}$| ≤** $\frac{1}{3}$ under the transformation w= $\frac{1}{z}$	CO 2	K2	57

4	Prove that $f(z) = e^z$ is conformal	CO 2	K5	41
5	Find the image of the following infinite strips under the mapping $w = \frac{1}{z}$ $\frac{1}{4} < y < \frac{1}{2}$	CO 2	K3	55
6	Find the fixed points of the bilinear transformation i) $w = \frac{z-1}{z+1}$ ii) $w = \frac{3z-4}{z-1}$	CO 2	K5	40
7	Find the image of the following infinite strips under the mapping $w = \frac{1}{z}$ $0 < y < \frac{1}{2}$	CO 2	K4	56
8	Discuss the conformality of f(z)= sinz	CO 2	K3	47
9	Explain the conformal mapping of f(z)= e^z	CO 2	K2	43
MODULE III				
1	Evaluate $\int_C Re\, z\, dz$, C is the shortest path from $1 + i$ to $3 + 3i$	CO 3	K4	78
2	Evaluate $\oint$ 3z dz $C: \|z - 4 - 2i\| = 5.5$	CO 3	K3	84
3	Find the Maclaurin series expansion of $f(z) = 1/_{1-z}$ and state the region of convergence.	CO 3	K2	100
4	Find the Taylor series of $f(z) = {}^{1/z}$ $about\ z = 2$	CO 3	K2	92
5	Find the Taylor series $f(z) = 1/z^2-z-6$ $about\ z = -1$	CO 3	K5	94

6	Evaluate ∮ e^z/ze^z -2iz *e* ______ ⍰⍰ ⍰: |⍰| = 0.6	CO 3	K4	97
7	1. Evaluate ∮ *dz/z-3i* ___ C is the circle $\|z\| = \pi$ counter clock wise	CO 3	K2	105
8	Find the Maclaurin series expansion of $f(z) = \frac{1}{1-⍰^2}$	CO 3	K3	102
MODULE IV				
1	Expand $f(z) = \frac{1}{z-⍰^3}$ in Laurent series for the region $1 < \|z+1\| < 2$	CO 4	K2	110
2	Find the residue of e^z at its pole. $\overline{3}$	CO 4	K1	107
3	Determine and classify the singularities of the function i) z/ $(z+1)^2(z+2)$ ii) ⍰−⍰i⍰⍰/z^3	CO 4	K2	120
4	Find the residues of $f(z) = \frac{50z}{⍰^3+2⍰^2-7⍰+4}$	CO 4	K3	115
5	Find all singular points and corresponding residues of $f(z)$= $(z+2)/(z+1)^2(z+2)$	CO 4	K1	130
6	Using contour integration evaluate ∫$^{2⍰}$ dΘ/ 2+cos⍰	CO 4	K2	123
7	Evaluate∮ $dz/(z^2+1)^2$ $where C: \|z-2-i\| = 3.2$	CO 4	K3	138
8	Find the Laurent series of $1/z^3\text{-}z^4$ $\overline{3\ \ 4}$ with Centre 0	CO 4	K3	109
9	Apply calculus of residues to evaluate	CO 4	K2	152

	$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}$			
MODULE V				
1	Check for consistence of the system $x+y+z=1,\quad x+2y+4z=2,\ x+4y+10z=4$	CO 5	K4	160
2	Show that the equations $3x+4y+5z=a,\ \ 4x+5y+6z=b,\ \ 5x+6y+7z=c$ do not have a solution unless $a+c=2b$	CO 5	K2	170
3	Reduce to Echelon form and hence find the rank of the Matrix $\begin{matrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & 5 \end{matrix}$	CO 5	K3	184
4	Find the rank $\begin{matrix} 0 & 1 & 0 \\ -1 & 0 & -4 \\ 0 & 4 & 0 \end{matrix}$	CO 5	K2	185
5	Using Gauss elimination method ,find the solution of the system of equations $x+2y-z=3, 3x-y+2z=1,\ 2x-2y+3z=2\ \ and\ \ x-y+z=-1.$	CO 5	K3	180
6	Find the values of μ for which the system of equations $x+y+z=1, x+2y+3z=\mu\ \ and$ $x+5y+9z=\mu^2$ will be consistant . For each value of μ obtained find the solution of the system.	CO 5	K2	190
7	Solve by Gauss elimination : $x_1-x_2+x_3=0\ ,-x_1+x_2-x_3=0\ ,\ 10x_2+25x_3=90,\ 20x_1+10x_2=80$	CO 5	K2	183

	MODULE VI			
1	Show that the eigen vectors of the symmetric matrix 1 1 3 1 5 1 3 1 1	CO 5	K3	196
2	Find out what type of conic section does follows quadratic form represents and transform it into principal axes if $Q = 4x_1^2 + 24x_1x_2 - 14x_2^2$=20	CO 5	K3	190
3	Is the matrix A is orthogonal if A = 1 -1 0 -1 1 0 1 1 1	CO 5	K2	188
4	Reduce Q = $x^2 + 3y^2 + 3z^2$ -$2yz$ into principal axis and find the canonical form.	CO 5	K5	184
5	Show that the matrix A = 1 2 2 -2 symmetric .Find Spectrum	CO 5	K2	199
6	The product of two eigen values of the matrix A= 6 -2 2 -2 3 -1 2 -1 3 Find the third eigen value	CO 5	K2	188
7	Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.	Co5	K3	190

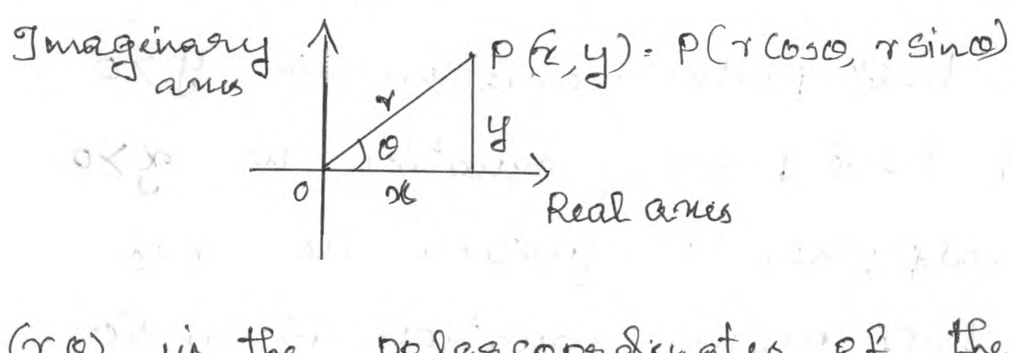
Complex Differentiation

MODULE - III

A complex no. 'z' is an ordered pair (x, y) of real no. written as $z = x + iy$, $i = \sqrt{-1}$, 'x' is the real part and 'y' is the imaginary part of z, ie $Re(z) = x$, $Im(z) = y$

Two complex no. are said to be equal if and only their real parts are equal and their imaginary parts are equal. The complex no. $\bar{z} = x - iy$ is called the complex conjugate of $z = x + iy$.

In Argon diagram, z is represented by a point $P(x, y)$.



If (r, θ) is the polarcoordinates of the

$x = r\cos\theta$, $y = r\sin\theta$.

then $r = \sqrt{x^2+y^2}$ Called the modulus of z,

ie $|z| = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called the augment denoted by arg z. Now $z = x+iy = r\cos\theta + i\sin\theta$

$$= r(\cos\theta + i\sin\theta)$$

$$\Rightarrow z = re^{i\theta}$$

Circle:

$|z-a| = r$ is the equation to a circle with centre at $(a,0)$ and radius r.

Eg: $|z-2| = 5$; centre at $(2,0)$ radius 5.

$|z+2| = 5$; centre at $(-2,0)$ radius 5.

$|z-2i| = 5$; centre at $(0,2)$ radius 5.

$|z+2i| = 5$; centre at $(0,-2)$ radius 5.

open circular disk equation is $|z-a| < r$.

closed circular disk ; equation is $|z-a| \le r$.

open annulus ; equation is $r_1 < |z-a| < r_2$

closed annulus ; equation is $r_1 \le |z-a| \le r_2$.

Upper half plane ; equation is $y > 0$

Lower half plane ; equation is $y < 0$

Left half plane ; equation is $x < 0$.

Right half plane ; equation is $x > 0$

3) Show that the function $f(x, y) = \frac{xy(x-y)}{x^2+y^2}$ is continous at the origin given that $f(0,0) = 0$.

Soln:

$$f(x,y) = \frac{xy\,(x-y)}{x^2+y^2}$$

along the path $y = mx$, $z \to 0 \Rightarrow x \to 0,\ y \to 0$

$$\therefore \lim_{z\to 0} \frac{xy\,(x-y)}{x^2+y^2} = \underset{x\to 0}{lt} \frac{x \cdot mx\,(x - mx)}{x^2 + m^2x^2}$$

$$= \underset{x\to 0}{lt} \frac{mx^3\,(1-m)}{x^2\,(1+m^2)}$$

$$= \underset{x\to 0}{lt}\ mx\,\frac{(1-m)}{(1+m^2)}$$

$= 0 \Rightarrow$ limit exist at $z=0$

Also $\underset{z\to 0}{lt} f(z) = f(0)$

clearly $\underset{z\to 0}{lt} \frac{xy\,(x-y)}{x^2+y^2} = 0 = f(0)$

$\therefore$ $f(z)$ is continous at $z = 0$.

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④ Show that $\underset{z\to 0}{lt} \frac{xy}{x^2+y^2}$ does not exist.

Soln:

$$\underset{z\to 0}{lt} \frac{xy}{x^2+y^2} = \underset{\substack{x\to 0\\ y\to 0}}{lt} \frac{x \cdot mx}{x^2+m^2x^2}$$

$$= \lim_{x \to 0} m \cdot \frac{x^2}{x^2(1+m^2)} = \lim_{x \to 0} \frac{m}{1+m^2}$$

which depends on 'm'.

∴ The limit is not unique.

∴ $\lim_{z \to 0} f(z)$ does not exist at $z=0$.

(5) Find out whether $f(z)$ is continous at $z=0$ if

$$f(z) = \begin{cases} \dfrac{Re(z^2)}{|z|} & \text{for } z \neq 0 \\ 0 & , z = 0 \end{cases}$$

<u>soln</u>:

$$f(z) = \frac{Re(z^2)}{|z|} = \frac{x^2-y^2}{\sqrt{x^2+y^2}}$$

Now $$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$$

along the path $y = mx$, $$\lim_{x \to 0} \frac{x^2 - m^2x^2}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x \to 0} \frac{x^2(1-m^2)}{x\sqrt{1+m^2}} = \lim_{x \to 0} \frac{x(1-m^2)}{\sqrt{1+m^2}}$$

$$= 0$$

∴ $\lim_{z \to 0} f(z) = 0$, exist.

To check continuity, $\lim_{z \to a} f(z) = f(a)$

ie $\lim_{z \to 0} f(z) = f(0) = 0$

∴ function is continous at $z=0$ //.

(6) Find out whether $f(z)$ is continous at $z=0$

if $f(z) = \begin{cases} \dfrac{Re(z)}{1+|z|} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

Soln:

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{Re(z)}{1+|z|}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{1+\sqrt{x^2+y^2}} = \lim_{x \to 0} \frac{x}{1+\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x \to 0} \frac{x}{1+x\sqrt{1+m^2}} = \frac{0}{1+0} = 0$$

$$= 0$$

$\therefore \lim_{z \to 0} f(z)$ exist

Now $f(0) = 0$

$\therefore \lim_{z \to 0} f(z) = f(0) = 0$

$\therefore$ function is continous at $z=0$.

Q) Check the continuity of $f(z) = \begin{cases} \dfrac{Re(z)}{1-|z|} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

at $z = 0$.

Soln:

$$f(z) = \frac{Re(z)}{1-|z|} = \frac{x}{1-\sqrt{x^2+y^2}}$$

$$\therefore \lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{Re(z)}{1-|z|} = \lim_{z\to 0} \frac{x}{1-\sqrt{x^2+y^2}}$$

along $y = mx$,

$$\lim_{z\to 0} \frac{x}{1-\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{1-\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x\to 0} \frac{x}{1-x\sqrt{1+m^2}} = \frac{0}{1} = \underline{\underline{0}}$$

$$\Rightarrow \lim_{z\to 0} \frac{Re(z)}{1-|z|} = 0$$

To check continuity of $f(z)$,

$$\lim_{z\to a} f(z) = f(a)$$

clearly $f(a) = f(0) = 0$.

ie $\lim_{z\to 0} f(z) = f(0) = 0$

$\therefore f(z) = \dfrac{Re(z)}{1-|z|}$ is continous at $z = 0$.

Q) check the continuity of $f(z) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

at $z=0$

Soln:

$$f(z) = \frac{x^2 y}{x^2+y^2} \qquad z = x+iy$$

$$\underset{z\to 0}{\text{lt}}\, f(z) = \underset{z\to 0}{\text{lt}}\, \frac{x^2 y}{x^2+y^2}$$

along the path $y = mx$

$$\underset{z\to 0}{\text{lt}}\, \frac{x^2 y}{x^2+y^2} = \underset{x\to 0}{\text{lt}}\, \frac{x^2 . mx}{x^2+m^2x^2} = \underset{x\to 0}{\text{lt}}\, \frac{x^3 m}{x^2(1+m^2)}$$

$$= \underset{x\to 0}{\text{lt}}\, x \frac{m}{1+m^2} = 0$$

ie $\underset{z\to 0}{\text{lt}}\, f(z) = \underline{\underline{0}}$

∴ Limit exist.

To check the continuity,

$$\underset{z\to a}{\text{lt}}\, f(z) = f(a)$$

$$\underset{z\to 0}{\text{lt}}\, f(z) = f(0)$$

$$\Rightarrow \quad 0 = 0.$$

$$\therefore f(a) = f(0) = 0$$

ely $\underset{z\to 0}{\text{lt}}\, f(z) = f(0) = 0$

∴ $f(z)$ is continous at $z=0$.

Q) check the continuity of $f(z) = \begin{cases} |z|^2 \, Im\left(\frac{1}{z}\right), & z \neq 0 \\ 0, & z = 0 \end{cases}$

at $z = 0$

Soln:

$$f(z) = |z|^2 \, Im\left(\frac{1}{z}\right)$$

$$= (x^2+y^2) \cdot Im\left(\frac{1}{x+iy}\right)$$

$$= (x^2+y^2) \cdot Im\left(\frac{x-iy}{(x-iy)(x+iy)}\right)$$

$$= (x^2+y^2) \cdot Im\left(\frac{x-iy}{x^2+y^2}\right)$$

$$= (x^2+y^2) \cdot \left(\frac{-y}{x^2+y^2}\right) = -y$$

$$f(z) = -y$$

Now $\underset{z \to 0}{lt} f(z) = \underset{z \to 0}{lt} -y$

along the path $y = mx$,

$$\underset{z \to 0}{lt} -y = \underset{x \to 0}{lt} -mx = 0$$

ie $\underset{z \to 0}{lt} f(z) = 0$ ∴ Limit exist.

~~clearly~~ To check the continuity,

$$\underset{z \to a}{lt} f(z) = f(a)$$

clearly $f(a) = f(0) = 0$

ie $\underset{z \to 0}{lt} f(z) = f(0) = 0$

∴ $f(z) = |z|^2 \, Im\left(\frac{1}{z}\right)$ is continous at $z = 0$.

DERIVATIVE OF ANALYTIC FUNCTIONS

If $w = f(z)$ a complex function, then the derivative of $f(z) = f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

Q) Show that $f(z) = z^2$ is differentiable for all z.

Proof:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$f(z) = z^2$

$f(z+\Delta z) = (z+\Delta z)^2$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + 2z \cdot \Delta z + (\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z\,\Delta z + (\Delta z)^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z \cdot \Delta z}{\Delta z} + \frac{(\Delta z)^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2z + \Delta z = 2z + 0$$

$$= 2z$$

ie $f'(z) = 2z$

$\therefore f(z) = z^2$ is differentiable for all z.

Q) check the differentiability of $f(z) = |z|^2$

<u>Soln</u>:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$f(z) = |z|^2$

$f(z+\Delta z) = |z+\Delta z|^2$

$$= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\cdot\bar{z}}{\Delta z}$$

$\therefore |z|^2 = z\cdot\bar{z}$

$|z+\Delta z|^2 = (z+\Delta z)\overline{(z+\Delta z)}$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\Delta\bar{z}) - z\cdot\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\cdot\bar{z} + z\cdot\Delta\bar{z} + \Delta z\cdot\bar{z} + \Delta z\cdot\Delta\bar{z} - z\cdot\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\cdot\Delta\bar{z} + \Delta z\cdot\bar{z} + \Delta z\cdot\Delta\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} z\cdot\frac{\Delta\bar{z}}{\Delta z} + \frac{\Delta z\cdot\bar{z}}{\Delta z} + \frac{\Delta z\cdot\Delta\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} z\cdot\frac{\Delta\bar{z}}{\Delta z} + \bar{z} + \Delta\bar{z}$$

Since $\lim_{\Delta z \to 0} z\cdot\frac{\Delta\bar{z}}{\Delta z}$ does not exist,

$\therefore f(z) = |z|^2$ is not differentiable any where.

[$\because \lim_{\Delta z \to 0} \frac{\Delta\bar{z}}{\Delta z}$ not exist.]

Q) check the differentiability of $f(z) = \bar{z}$.

Soln:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \Delta\bar{z} - \bar{z}}{\Delta z} \qquad \because \overline{z+\Delta z} = \bar{z} + \overline{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z}$$

since $\lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z}$ does not exist, $f(z) = \bar{z}$ is not differentiable any where.

ANALYTIC FUNCTIONS

A function is said to be analytic in a domain D if f(z) is defined and differentiable at all points of D.

CAUCHY - REIMANN EQUATION [C-R eqn]

Let $f(z) = u(x,y) + i\,v(x,y)$ is analytic in domain D if its partial derivative exists and satisfies the conditions

$$\boxed{\begin{aligned} U_x &= V_y \quad \text{and} \\ U_y &= -V_x \end{aligned}}$$

which is called C-R equation

Q) Show that $f(z) = z^2$ is analytic for all z.

Soln:-

$f(z) = z^2$ is analytic if Cauchy Reimann equations (C-R) are satisfied $\Rightarrow$ $U_x = V_y$

$U_y = -V_x$

Now $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i\,2xy$ — ①

$U_x = \frac{\partial u}{\partial x}$, $U_y = \frac{\partial u}{\partial y}$, $V_x = \frac{\partial v}{\partial x}$ & $V_y = \frac{\partial v}{\partial y}$

from ① $\Rightarrow$ $U = U(x,y) = Re(z^2) = x^2 - y^2$

$\Rightarrow U = x^2 - y^2$

also $\Rightarrow$ $V = ~~2x~~ V(x,y) = Im(z^2) = 2xy$

$\Rightarrow V = 2xy$

$\therefore U_x = \frac{\partial u}{\partial x} = 2x$. $V_x = \frac{\partial v}{\partial x} = 2y$

$U_y = \frac{\partial u}{\partial y} = -2y$ $V_y = \frac{\partial v}{\partial y} = 2x$

~~clearly $U_x = 2x =$~~

C-R eqn is $U_x = V_y$

$U_y = -V_x$.

clearly $U_x = 2x = V_y$

and $U_y = -2y = -V_x$ $\Rightarrow$ C-R eqn are satisfied.

$\therefore f(z) = z^2$ is ~~not~~ analytic

Q) Show that $f(z) = e^z$ is analytic everywhere.

PROOF :

Cauchy - Riemann equations are,

$U_x = V_y$ and $U_y = -V_x$

Now $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$

$= e^x [\cos y + i \sin y]$

Since, $e^{ix} = \cos x + i \sin x$

ie $f(z) = e^x \cos y + i e^x \sin y$

$f(z) = U(x,y) + i V(x,y)$

Where $U = U(x,y) = e^x \cos y$

$V = V(x,y) = e^x \sin y$

$U_x = \frac{\partial u}{\partial x} = e^x \cos y$

$U_y = \frac{\partial u}{\partial y} = -e^x \sin y$

$V_x = \frac{\partial v}{\partial x} = e^x \sin y$

$V_y = \frac{\partial v}{\partial y} = e^x \cos y$

clearly $U_x = V_y = e^x \cos y$ and

$U_y = -V_x = -e^x \sin y$.

C-R equations are satisfied

$\therefore f(z) = e^z$ is analytic everywhere.

Q) Test the analyticity of $f(z) = Re(z^2) - Im(z^2)$

Soln:

C-R equation is $U_x = V_y$ and $U_y = -V_x$

$f(z) = Re(z^2) - Im(z^2)$

$= Re\,(x+iy)^2 - Im\,(x+iy)^2$

$= Re\,(x^2 - y^2 + i2xy) - Im\,(x^2 - y^2 + i2xy)$

$= x^2 - y^2 - 2xy$

$z^2 = (x+iy)^2$
$z^2 = x^2 - y^2 + i2xy$
$Re(z^2) = x^2 - y^2$
$Im(z^2) = 2xy$.

ie $f(z) = x^2 - y^2 - 2xy$ — (1)

$\Rightarrow U = x^2 - y^2 - 2xy$.

Now $U_x = \frac{\partial u}{\partial x} = 2x - 2y$

$U_y = \frac{\partial u}{\partial y} = -2y - 2x = -(2x + 2y)$

from (1), $V = 0$

$\therefore V_y = 0$ and $V_x = 0$.

C-R equns are not satisfied

ie $U_x \neq V_y$ and $U_y \neq -V_x$

$\therefore f(z) = Re(z^2) - Im(z^2)$ is not analytic.

Q) Show that $f(z) = \sin z$ is analytic everywhere.

Soln:

C-R eqn is $U_x = V_y$ and $U_y = -V_x$

$f(z) = \sin z = \sin(x+iy)$

$= \sin x(\cos iy) + \cos x \sin(iy)$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\sin(iy) = i\sinh y$ $\cos(iy) = \cosh y$

$\Rightarrow f(z) = \sin x \cosh y + \cos x \, i \sinh y$

$\Rightarrow f(z) = \sin x \cosh y + i \cos x \sinh y$

$= U(x,y) + i\, V(x,y)$

$\therefore U = \sin x \cosh y$, $V = \cos x \sinh y$

$\frac{d}{dy}\cosh y = \sinh y$ $\frac{d}{dy}\sinh y = \cosh y$

$U_x = \frac{\partial u}{\partial x} = \cos x \cosh y$

$U_y = \frac{\partial u}{\partial y} = \sin x \sinh y$

$V_x = \frac{\partial v}{\partial x} = -\sin x \sinh y$

$V_y = \frac{\partial v}{\partial y} = \cos x \cosh y$

clearly, $U_x = V_y = \cos x \cosh y$

and $U_y = -V_x = -(-\sin x \sinh y)$

$\therefore$ C-R eqns are satisfied.

$\therefore f(z) = \sin z$ is analytic everywhere.

Q) Test the analyticity of $w = \cos z$.

Soln:-

C-R eqns are $U_x = V_y$ and $U_y = -V_x$

$f(z) = \cos z = \cos(x+iy)$

$= \cos x \cos(iy) - \sin x \sin(iy)$

$= \cos x \cosh y - i \sin x \sinh y$

$= U(x,y) + i V(x,y)$

$\cos(iy) = \cosh y$
$\sin(iy) = i \sinh y$

Where $U = U(x,y) = \cos x \cosh y$

$V = V(x,y) = -\sin x \sinh y$

Result: $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$U_x = \frac{\partial u}{\partial x} = -\sin x \cosh y$

$U_y = \frac{\partial u}{\partial y} = \cos x \sinh y$

$\frac{d}{dy} \sinh y = \cosh y$
$\frac{d}{dy} \cosh y = \sinh y$

$V_x = \frac{\partial v}{\partial x} = -\cos x \sinh y$

$V_y = \frac{\partial v}{\partial y} = -\sin x \cosh y$.

Clearly, $U_x = V_y = -\sin x \cosh y$

$U_y = \cos x \sinh y = -V_x$.

C-R eqns are satisfied.

$\therefore f(z) = \cos z$ is analytic everywhere.

LAPLACE EQUATION

If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D, then u and v satisfy Laplace eqn.

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

and $\nabla^2 v = v_{xx} + v_{yy} = 0$

Note:- ① Solutions of laplace equation having continous 2nd order partial derivatives are called harmonic functions and that theory is called potential theory.

② The real and imaginary parts of analytic functions are harmonic functions.

③ If two harmonic functions, u and v satisfy C-R equations, the real and imaginary parts are analytic functions.

Q) Verify that $u(x,y) = x^2 - y^2$ is harmonic and find its harmonic conjugate. Also give the associated analytic function.

Soln:

Laplace equation is $u_{xx} + u_{yy} = 0$ or

$U = x^2 - y^2$

$U_x = 2x$, $U_y = -2y$

$U_{xx} = 2$ $U_{yy} = -2$

$\therefore U_{xx} + U_{yy} = 2 - 2 = 0.$

Laplace equation is satisfied.

$\therefore$ It is harmonic function.

To find harmonic conjugate, take C-R equation

$U_x = V_y$ and $U_y = -V_x$.

Now $U_x = V_y \Rightarrow 2x = V_y$.

Integrating both sides w.r.to y, we get

$$\int 2x \, dy = \int V_y \, dy$$

$$\Rightarrow 2xy + h(x) = V$$

$$\Rightarrow V = 2xy + h(x) \text{ --- (1)}$$

To find $h(x)$, differentiate w.r.to x, we get

$$\Rightarrow V_x = 2y + h'(x) = -U_y$$ ($\because$ using C-R $-V_x = U_y$.)

$$\Rightarrow \quad = 2y + h'(x) = -(-2y)$$

$$\Rightarrow h'(x) = 2y - 2y = 0$$

$$\Rightarrow h'(x) = 0$$

$$\Rightarrow h(x) = c \text{, a constant.}$$

Substitute $h(x) = c$ in eqn (1), we get

$$V = 2xy + c$$

$\therefore$ Analytic function is $f(z) = u + iv$

$$\Rightarrow f(z) = (x^2 - y^2) + i(2xy + c)$$

Q) Show that $u = x^2 - y^2 - y$ is harmonic, find the harmonic conjugate & the corresponding analytic function.

Soln:

Given $u = x^2 - y^2 - y$

$u_x = 2x \qquad u_y = -2y - 1 = -(2y+1)$

$u_{xx} = 2 \qquad u_{yy} = -2$

$\therefore u_{xx} + u_{yy} = 0 \quad$ ie $u_{xx} + u_{yy} = -2 + 2 = 0$

$\therefore$ u satisfies the Laplace equation.

$\therefore$ u is harmonic

f(z) is analytic $\Rightarrow$ C-R equations are satisfied.

$\therefore \quad u_x = v_y \quad$ and $\quad u_y = -v_x$

$u_x = v_y \Rightarrow 2x = v_y$, To find harmonic Conjugate, Integrating both sides w.r.to y, we get

$$\int 2x \, dy = \int v_y \, dy$$

$$\Rightarrow 2xy + h(x) = v$$

$$\text{ie } v = 2xy + h(x) \quad \text{——— (1)}$$

differentiating both sides w.r.to x, we get

$$v_x = 2y + h'(x) = -u_y$$

C-R eqn, $u_y = -v_x$.

$$\text{ie } v_x = 2y + h'(x) = -(-(2y+1))$$

$$\Rightarrow 2y + h'(x) = 2y + 1$$

$$\Rightarrow h'(x) = 1 \quad \Rightarrow h(x) = x$$

$\therefore$ from ①, $V = 2xy + x$

Harmonic conjugate of u is $V = 2xy + x$

$\therefore$ Analytic function, $f(z) = u + iv$

ie $f(z) = (x^2 - y^2 - y) + i(2xy + x)$

Q) Prove that the function $u(x,y) = x^3 - 3xy^2 - 5y$ is harmonic everywhere. Also find harmonic conjugate of u?

Soln:-

$u = x^3 - 3xy^2 - 5y$

$u_x = 3x^2 - 3y^2$ $\quad\quad$ $u_y = -6xy - 5$

$u_{xx} = 6x$ $\quad\quad$ $u_{yy} = -6x$

$\therefore u_{xx} + u_{yy} = 6x + -6x = 0$

$\therefore$ u Satisfies Laplace equation.

$\therefore$ u is harmonic.

$\therefore$ $f(z) = u + iv$ is analytic $\Rightarrow$ C-R equations are satisfied.

$u_x = V_y$ and $u_y = -V_x$

To find harmonic conjugate of u, take 1st C-R eqn.

$u_x = V_y \Rightarrow 3x^2 - 3y^2 = V_y$

Integrating both sides w.r.to y, we get.

$$\int (3x^2 - 3y^2)\, dy = \int V_y\, dy.$$

$\Rightarrow \quad V = 3x^2y - \frac{3y^3}{3} + h(x)$

$\Rightarrow \quad V = 3x^2y - y^3 + h(x)$ ——— ①

To find $h(x)$, differentiate both sides w.r.to x,

we get, $V_x = 6xy + h'(x) = -u_y$ (using C-R eqn, $u_y = -V_x$)

$\Rightarrow \quad V_x = 6xy + h'(x) = -(-6xy-5)$

$\Rightarrow \quad 6xy + h'(x) = -(-(6xy+5))$

$\Rightarrow \quad 6xy + h'(x) = 6xy + 5$

$\Rightarrow \quad h'(x) = 5$

$\Rightarrow \quad h(x) = 5x$

$\therefore$ from ①, $V = 3x^2y - y^3 + 5x$

$\therefore$ Harmonic Conjugate of u is $V = 3x^2y - y^3 + 5x$

Analytic function, $f(z) = u + iv$

ie $f(z) = (x^3 - 3xy^2 - 5y) + i(3x^2y - y^3 + 5x)$

Q) Determine a & b so that the function, $u = ax^3 + bxy$ is harmonic & find harmonic Conjugate.

Soln:

$u = ax^3 + bxy$

$u_x = 3ax^2 + by \qquad u_y = bx$

$u_{xx} = 6ax \qquad u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0$ [Since $u = ax^3 + bxy$ is harmonic. (given).

ie $6ax + 0 = 0$

$\Rightarrow 6ax = 0$

$\Rightarrow a = 0 \Rightarrow b$ is any non-zero real no:

$\therefore u = ax^3 + bxy$

$\Rightarrow u = bxy.$ $\left[\because a = 0 \right.$

To find harmonic conjugate of u, take C-R equations, $U_x = V_y$ and $U_y = -V_x$

$U_x = V_y \Rightarrow 3ax^2 + by = V_y.$

$\Rightarrow 0 + by = V_y$

Integrating both sides w.r.to y, we get

$$\int by\,dy = \int V_y\,dy$$

$$\Rightarrow \frac{by^2}{2} + h(x) = V$$

$$\Rightarrow V = \frac{by^2}{2} + h(x) \quad \text{——} \quad (1)$$

To find $h(x)$,
differentiating both sides w.r.to x,

ie $V_x = h'(x) = -u_y$ [Using C-R eqn, $u_y = -V_x$

$\Rightarrow h'(x) = -bx$

$\Rightarrow h(x) = \frac{-bx^2}{2}$

Substitute $h(x) = -\frac{bx^2}{2}$ in (1), we get.

$① \Rightarrow V = \frac{by^2}{2} + h(x)$

ie $V = \frac{by^2}{2} + -\frac{bx^2}{2}$ is the harmonic conjugate of u.

$\therefore$ Analytic function, $f(z) = u + iV$

$$\Rightarrow f(z) = (ax^3 + bxy) + i\left(\frac{by^2}{2} - \frac{bx^2}{2}\right)$$

$$\text{ie } f(z) = bxy + i\frac{b}{2}(y^2 - x^2)$$

Q) Determine 'a' and 'b' so that function $u = e^{-\pi x}\cos ay$ is harmonic. Find its harmonic conjugate?

Soln:

function is harmonic, $u = e^{-\pi x}\cos ay$

$\therefore u_{xx} + u_{yy} = 0$

$u_x = -\pi e^{-\pi x}\cos ay$

$u_{xx} = \pi^2 e^{-\pi x}\cos ay$

$u_y = e^{-\pi x} - a\sin ay$

$u_{yy} = e^{-\pi x} - a^2\cos ay.$

$u_{xx} + u_{yy} = 0 \Rightarrow \pi^2 e^{-\pi x}\cos ay + -a^2 e^{-\pi x}\cos ay = 0$

$\Rightarrow e^{-\pi x}\cos ay(\pi^2 - a^2) = 0$

$\Rightarrow \pi^2 = a^2$

$\Rightarrow \pi = \pm a.$

$\therefore u = e^{-ax}\cos ay.$

To find harmonic conjugate of u,

take C-R eqns, $u_x = V_y.$

ie $-\pi e^{-\pi x}\cos ay = V_y.$

Integrating both sides w.r.to y, we get

$$\int v_y \, dy = \int -ae^{-ax} \cos ay \, dy \quad \angle \therefore \pi = a.$$

$$v = -ae^{-ax} \frac{\sin ay}{a} + h(x)$$

ie $v = -e^{-ax} \sin ay + h(x)$ — ①

differentiating ① w.r.to x,

$$v_x = ae^{-ax} \sin ay + h'(x) = -u_y \quad \angle \therefore \text{CR eqn.}$$

ie $ae^{-ax} \sin ay + h'(x) = -(e^{-ax} - a \sin ay)$

$$\Rightarrow ae^{-ax} \sin ay + h'(x) = ae^{-ax} \sin ay$$

$$\Rightarrow h'(x) = 0.$$

$$\Rightarrow h(x) = c \text{, a constant.}$$

$\therefore$ ① $\Rightarrow$ $v = -e^{-ax} \sin ay + c$ is harmonic conjugate of u.

MODULE - III

CONFORMAL MAPPING

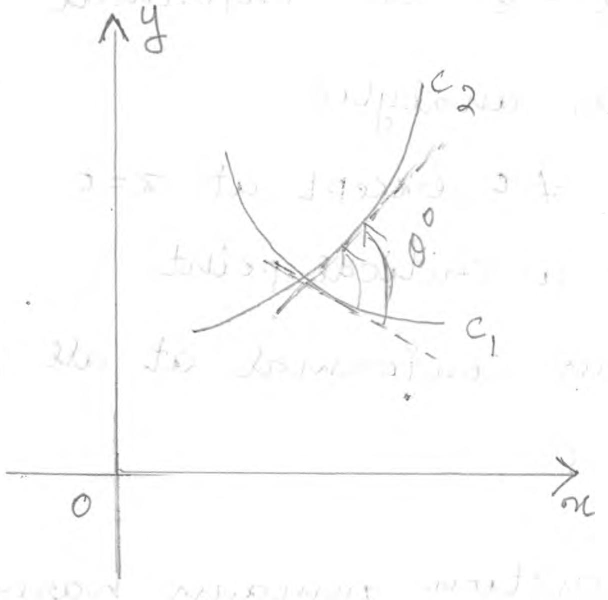
The complex function $w = f(z)$

ie $u+iv = f(x+iy)$

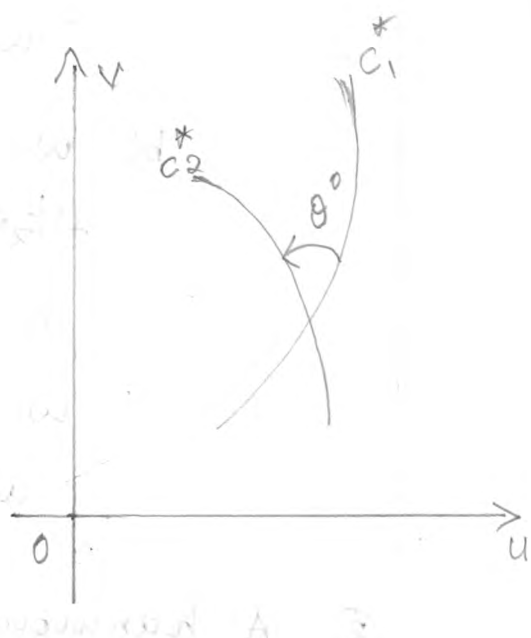
involves 4 variables, 2 independent variables x and y and 2 dependent variables u and v.

A 4-dimensional region is required to represent it graphically. As it is not possible, we choose two complex planes z-plane and w-plane.

In the z-plane we plot $z = x+iy$ and in the w-plane we plot $w = u+iv$. So the function $w = f(z)$ defines a correspondance between points of these two planes.



z-plane



w-plane.

A transformation which preserves magnitude and sense of the angle between every pair of curves in some domain in the z-plane as its image in the w-plane is called conformal mapping.

A transformation which preserves angles in magnitude alone is called isogonal mapping.

Note:

① If $w = f(z)$ is analytic then it is conformal at each points of its domain, provided $f'(z) \neq 0$. The points at which $f'(z) = 0$, are called critical points.

Eg: a) $w = e^z$ is analytic

$f'(z) = e^z \neq 0$ for all z.

$\therefore w = f(z) = e^z$ is Conformal.

b) $w = z^2$ is analytic.

$f'(z) = 2z \neq 0$ except at $z = 0$

$z = 0$ is a critical point

$\therefore w = z^2$ is Conformal at all points except at $z = 0$.

② A harmonic function remains harmonic under a Conformal transformation.

Q) Discuss the mapping or transformation of $w = z^2$

Soln:

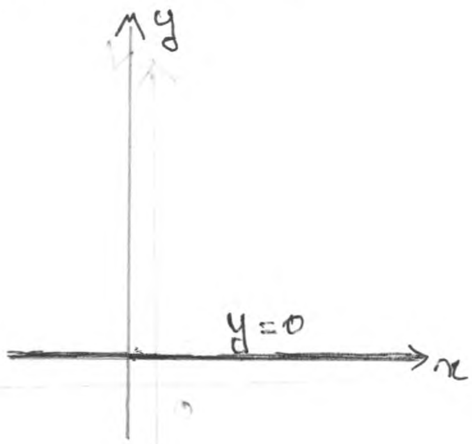
$$w = z^2$$

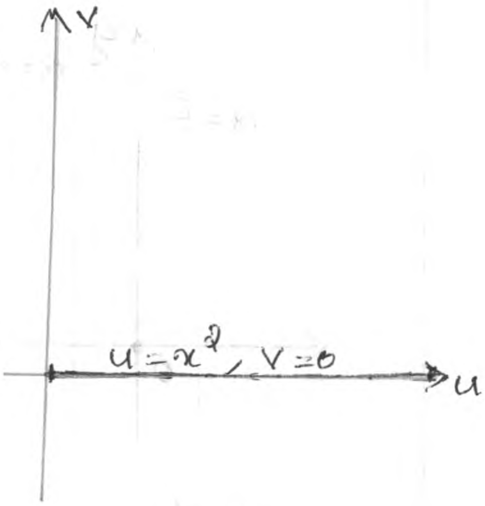
$$u + iv = (x+iy)^2 = x^2 - y^2 + i\,2xy$$

$$\Rightarrow u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

Case 1:

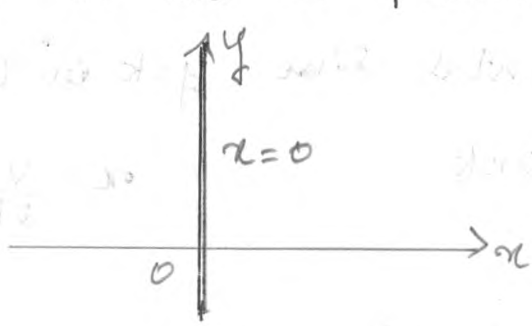
Consider the real axis $y=0$ in the z-plane. then $u = x^2$, $v=0$; which is the positive real axis in the w-plane.



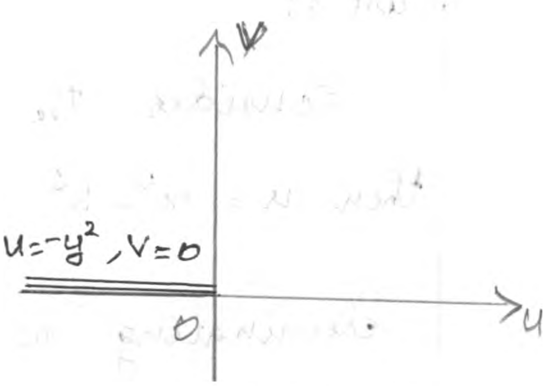


Case 2:-

Consider the imaginary axis $x=0$ in the z-plane then $u = -y^2$, $v=0$; which is the negative real axis in the w-plane.



z-plane

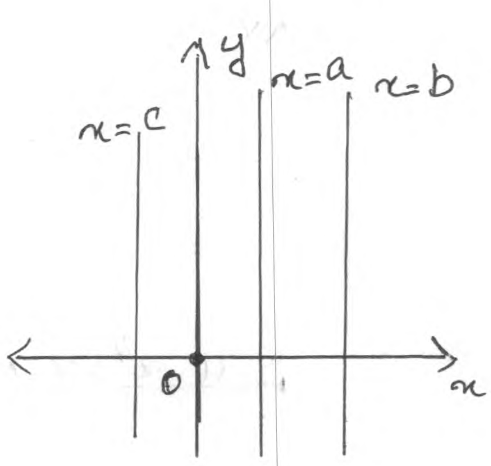


w-plane.

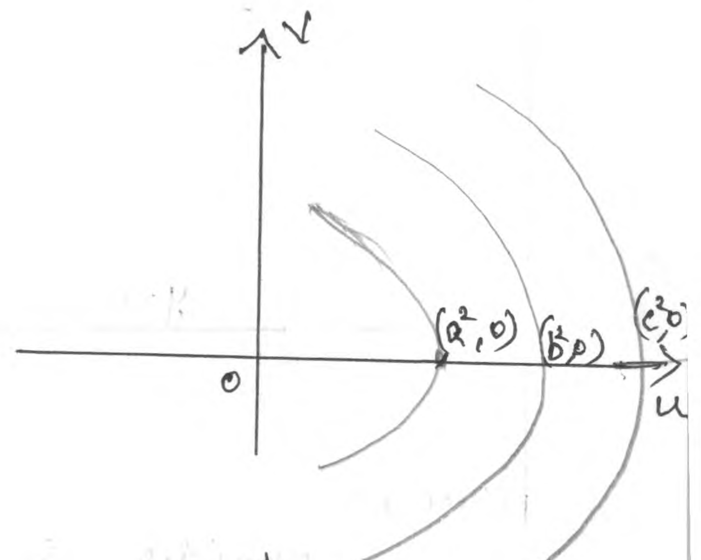
Case 3:- Consider the vertical line $x = c$ in the z-plane then $u = c^2 - y^2$; $v = 2cy$.

eliminating y,
$$u = c^2 - y^2$$
$$u = c^2 - \left(\frac{v}{2c}\right)^2 \qquad \therefore\ y = \frac{v}{2c}$$
$$\frac{v^2}{4c^2} = c^2 - u$$
$$v^2 = 4c^2(c^2 - u)$$
$$(v-0)^2 = -4c^2(u - c^2),$$
which is a left handed parabola with vertex at $(x_1, y_1) = (c^2, 0)$.

$(y - y_1)^2 = 4a(x - x_1)$
Vertex at $(x_1, y_1) \Rightarrow$ parabola



z-plane.



w-plane.

Case 4:

Consider the horizontal line $y = k$ in the z-plane.
then $u = x^2 - k^2$, $v = 2xk$ $\qquad \therefore\ x = \frac{v}{2k}$

eliminating x,
$$u = x^2 - k^2$$
$$\Rightarrow\ u = x^2 - \left(\frac{v}{2k}\right)^2$$

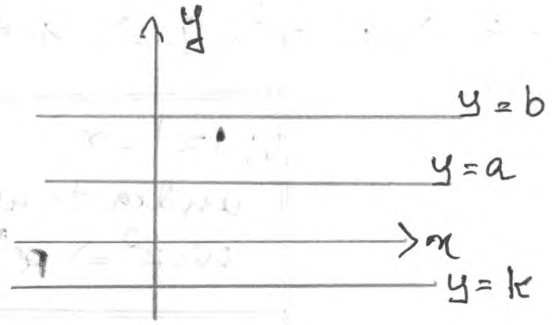
$$\Rightarrow u = \left(\frac{v}{2k}\right)^2 - k^2$$

$$\frac{v^2}{4k^2} = u + k^2$$

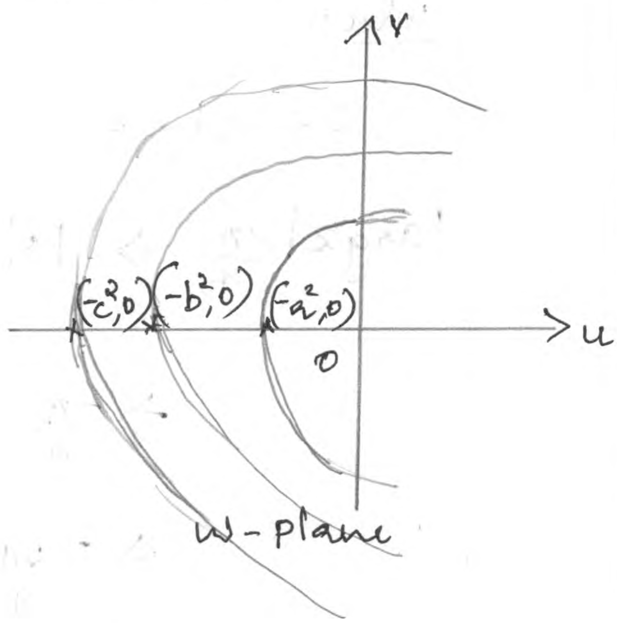
$$v^2 = 4k^2(u + k^2)$$

$$(v - 0)^2 = 4k^2(u - -k^2)$$

which is a right handed parabola with vertex at $(-k^2, 0)$



z-plane



w-plane

Also $f'(z) = 2z \neq 0$ except at $z = 0$

Hence the transformation $w = z^2$ is conformal at all points except at $z = 0$.

REMARK

Using polar forms $z = re^{i\theta}$ in the z-plane, $w = Re^{i\phi}$ in the w-plane.

$$w = z^2 \Rightarrow Re^{i\phi} = (re^{i\theta})^2$$

$$\Rightarrow Re^{i\phi} = r^2 e^{i2\theta}$$

$$\Rightarrow R = r^2 \text{ and } \phi = 2\theta.$$

Hence the circle having radius r_0 is mapped to circle having radius r_0^2 and θ_0 mapped to $2\theta_0$

Q) Find the image of the region $2 < |z| < 3$; $|\arg z| < \frac{\pi}{4}$ under the map $w = z^2$.

Soln:-

$$2 < |z| < 3 \Rightarrow 2 < r < 3 \Rightarrow 4 < r^2 < 9 \Rightarrow 4 < R < 9$$

$$|\arg z| < \frac{\pi}{4} \Rightarrow |\theta| < \frac{\pi}{4}$$

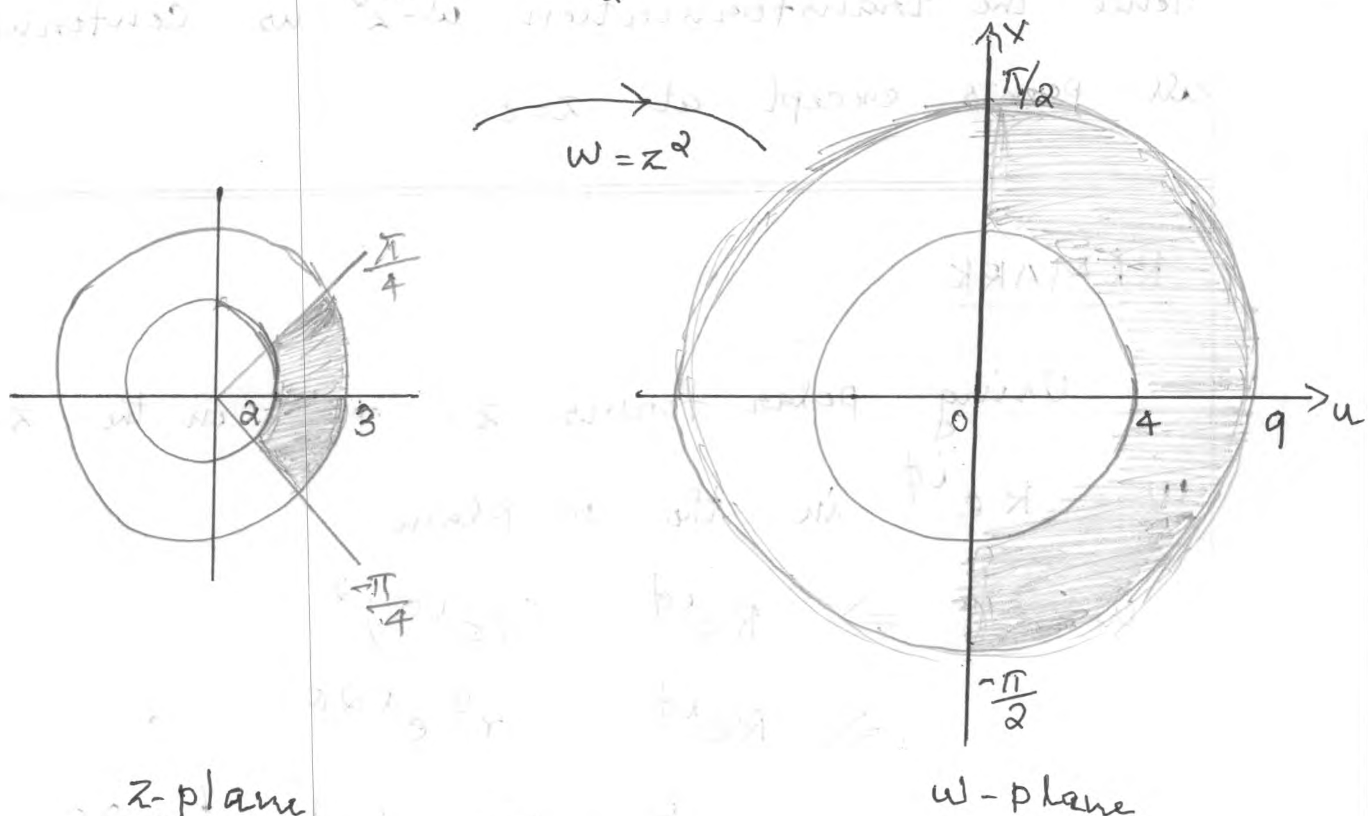
$$\Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

$$\Rightarrow -\frac{2\pi}{4} < 2\theta < \frac{2\pi}{4}$$

$$\Rightarrow -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

∵ $|z| = r$ under transformation $w = z^2 \Rightarrow R = r^2$ & $\phi = 2\theta$

∴ $\arg z = \theta$ under $w = z^2$, $\phi = 2\theta$.



✗. MAPPING OF $w = e^z$

Q. Conformal mapping of $w = e^z$.

Soln:

$$w = e^z \Rightarrow u + iv = e^{x+iy} = e^x \cdot e^{iy}$$

$$\Rightarrow u + iv = e^x[\cos y + i \sin y]$$

$$\Rightarrow u = e^x \cos y, \quad v = e^x \sin y.$$

Case 1:

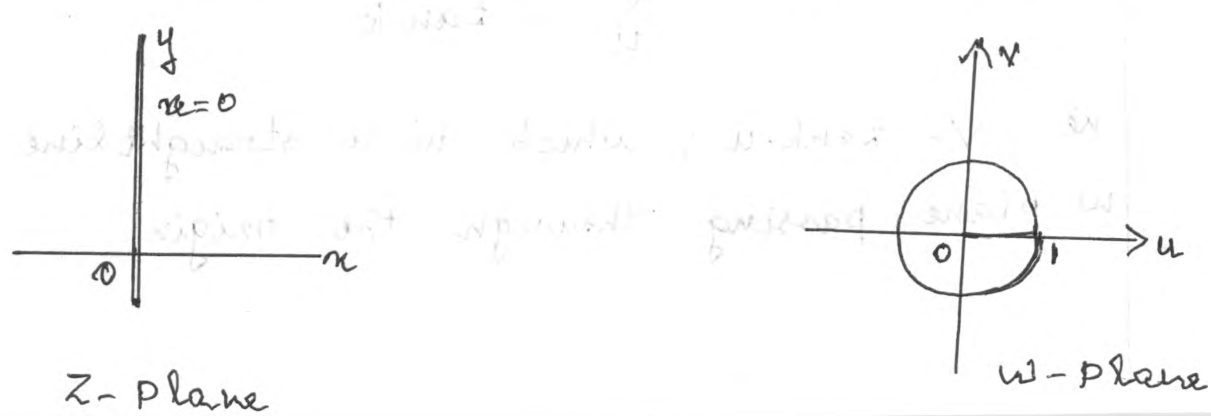
Consider the real axis $y = 0$ in the z-plane then $u = e^x$; $v = 0$ which is positive real axis in the w-plane.

Case 2 :-

Consider the imaginary axis $x = 0$ in the z-plane then $u = \cos y$; $v = \sin y$.

Eliminating y, $\cos^2 y + \sin^2 y = u^2 + v^2$

ie $u^2 + v^2 = 1$, which is a circle in the w-plane with centre at $(0,0)$ & radius 1.



Case 3:

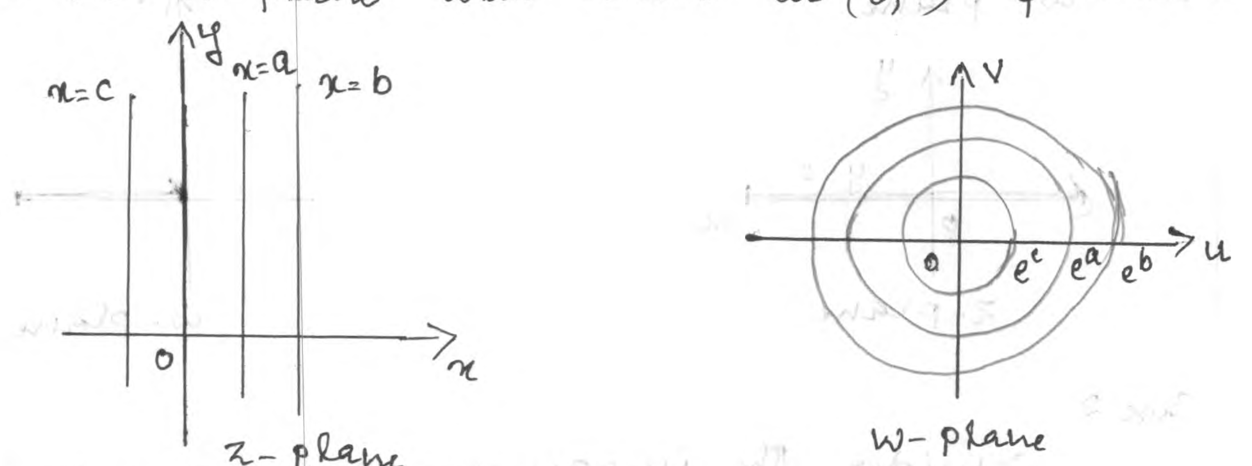
Consider the vertical line $x=c$ in the z-plane, then $u = e^c \cos y$; $v = e^c \sin y$

Eliminating y,

$$\cos^2 y + \sin^2 y = \left(\frac{u}{e^c}\right)^2 + \left(\frac{v}{e^c}\right)^2$$

$$\text{ie} \quad 1 = \frac{u^2}{(e^c)^2} + \frac{v^2}{(e^c)^2}$$

ie $u^2 + v^2 = (e^c)^2$ which is a circle in the w-plane with centre at $(0,0)$ & radius e^c.



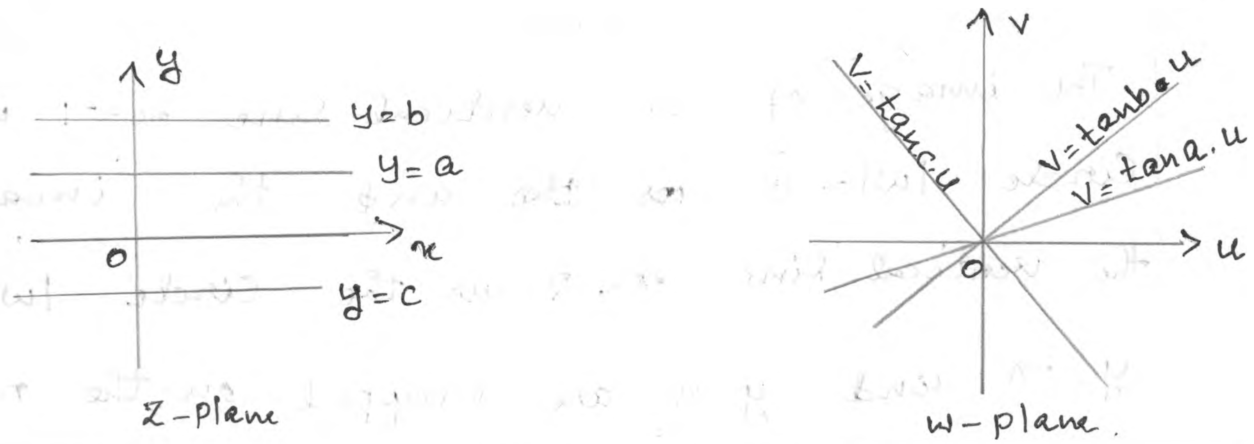
Case 4:-

Consider the horizontal lines $y=k$ in the z-plane then $u = e^x \cos k$; $v = e^x \sin k$

Eliminating x,

$$\frac{v}{u} = \tan k$$

ie $v = \tan k \cdot u$; which is a straight line in the w-plane passing through the origin.



Q) Find the image of the region $-\log 2 \le x \le \log 4$ under the mapping $w = e^z$.

Soln:

$w = e^z$

$-\log 2 \le x \le \log 4$

Image of the vertical line $x = -\ln 2$ is the circle $|w| = e^{-\ln 2}$ $\qquad e^{\log x} = x$

$\Rightarrow |w| = e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}$

Similarly, the image of the line $x = \ln 4$ is the circle $|w| = e^{\ln 4} \Rightarrow |w| = 4$.

Hence the region bounded by the lines $|w| = \frac{1}{2}$ and $|w| = 4$.

Q) Find the image of the region $-1 \le x \le 2$, $-\pi \le y \le \pi$ under the mapping $w = e^z$.

Soln: $-1 \le x \le 2 \Rightarrow x = -1, \; x = 2$

$-\pi \le y \le \pi \Rightarrow y = -\pi, \; y = \pi$.

The image of the vertical line $x=-1$ is the Circle $|w| = e^{-1}$ ~~is the~~ and the image of the vertical line $x = 2$ is the Circle $|w| = e^{2}$.

$y = -\pi$ and $y = \pi$ are mapped on the rays $\arg w = -\pi$ and $\arg w = \pi$ respectively.

Q) Mapping of $w = \frac{1}{z}$

Soln:-

$$u+iv = \frac{1}{x+iy}$$

$$\Rightarrow u+iv = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{-y}{x^2+y^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = \frac{-v}{u^2+v^2}$$

Q) Find the image of the following regions under the mapping $w = \frac{1}{z}$.

1) $\frac{1}{4} < y < \frac{1}{2}$

2) $0 < y < \frac{1}{2}$

3) $|z - 2i| = 2$.

Soln:

(1) $\frac{1}{4} < y < \frac{1}{2}$.

$\frac{1}{4} < y$ and $y < \frac{1}{2}$

Let $w = \frac{1}{z} = u + iv$

Then $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$

$\frac{1}{4} < y \Rightarrow \frac{1}{4} < \frac{-v}{u^2+v^2}$

$\Rightarrow u^2+v^2 < -4v$

$\Rightarrow u^2+v^2+4v < 0$ which is the interior part of the circle.

Now $y < \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2}$

$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2 > -2v$

$\Rightarrow u^2+v^2+2v > 0$ which is the exterior part of the circle.

$\therefore$ The image of the region $\frac{1}{4} < y < \frac{1}{2}$ is mapped in to the region between the circles $u^2+v^2+4v = 0$ and $u^2+v^2+2v = 0$.

(2) $0 < y < \frac{1}{2}$

Let $w = \frac{1}{z} = u + iv$

$\Rightarrow x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

$0 < y \quad \Rightarrow \quad 0 < \frac{-v}{u^2+v^2}$

$\Rightarrow -v > 0 \Rightarrow v > 0$ ∴ The image of the $0 < y$ is $v > 0$, which is the upper half pla

Now $y < \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2}$

$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2+2v > 0$, which is the exterior part of the circle.

∴ $0 < y < \frac{1}{2} \Rightarrow$ Images of $0 < y < \frac{1}{2}$ is mapped out $v > 0$ and $u^2+v^2+2v > 0$

(3) $|z-2i| = 2$

Let $w = \frac{1}{z} = u+iv$

$\Rightarrow x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$

Now $|z-2i| = 2 \Rightarrow |x+iy-2i| = 2$

$\Rightarrow |x+i(y-2)| = 2$

$\Rightarrow \sqrt{x^2+(y-2)^2} = 2$ ∴ $|x+iy| = \sqrt{x^2+y^2}$

Squaring both sides,

$\Rightarrow x^2+(y-2)^2 = 4$

$\Rightarrow x^2+y^2-4y+4 = 4$

3) Find the image of $|z-2i|=2$ under $w=\frac{1}{z}$.

Ans: $w=\frac{1}{z} \Rightarrow z=\frac{1}{w}$

$$\Rightarrow x+iy = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)}$$

$$x+iy = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

Equating real and imaginary parts,

$$x = \frac{u}{u^2+v^2}, \qquad y = \frac{-v}{u^2+v^2}$$

given $|z-2i|=2 \Rightarrow |x+iy-2i|=2 \Rightarrow |x+i(y-2)|=2$

$$\Rightarrow \sqrt{x^2+(y-2)^2} = 2$$

$$\Rightarrow x^2+(y-2)^2 = 4 = 2^2$$

$\Rightarrow$ circle with centre $(0,2)$ and radius 2.

ie $x^2+(y-2)^2 = 4$

$$\Rightarrow \left[\frac{u}{u^2+v^2}\right]^2 + \left[\left(\frac{-v}{u^2+v^2}\right)-2\right]^2 = 4$$

$$= \frac{u^2}{(u^2+v^2)^2} + \left(\frac{v}{u^2+v^2}+2\right)^2 = 4$$

$$= \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} + 4 = 4$$

$$= \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0$$

$$= \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0$$

$$= \frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

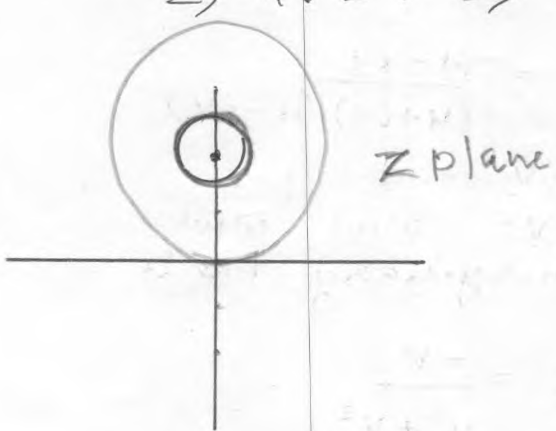
$$= (u^2+v^2) + 4v(u^2+v^2) = 0$$

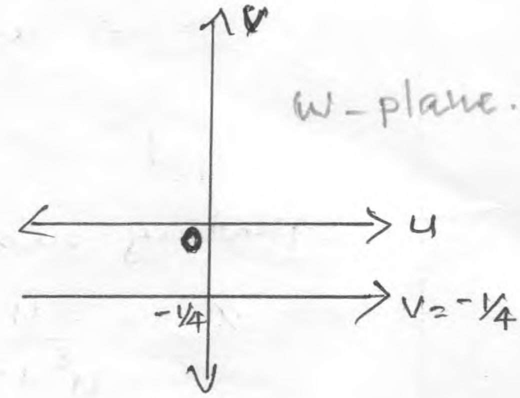
$$= u^2+v^2(1+4v) = 0$$

$\Rightarrow u^2+v^2 \neq 0$

$\therefore 4v+1 = 0$

$\Rightarrow 4v = -1 \Rightarrow v = -\frac{1}{4}$ is straight line.





4) Find the image of $|z+1|=1$ under the mapping $w=\frac{1}{z}$.

Ans: $z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

equating real and imaginary parts,

$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$

given $|z+1|=1$.

To find image of $|z+1|=1$,

$|z+1|=1 \Rightarrow |x+iy+1|=1$

$= |(x+1)+i(y)|=1$

$= \sqrt{(x+1)^2+y^2} = 1$

$= (x+1)^2+y^2 = 1^2=1$

$\Rightarrow$ centre $(-1,0)$ and radius 1.

$(x+1)^2+y^2=1 \Rightarrow \left[\frac{u}{u^2+v^2}+1\right]^2 + \left[\frac{-v}{u^2+v^2}\right]^2 = 1$

$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{2u}{u^2+v^2} + 1 + \frac{v^2}{(u^2+v^2)^2} = 1$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{2u}{u^2+v^2} + 1 = 1$$

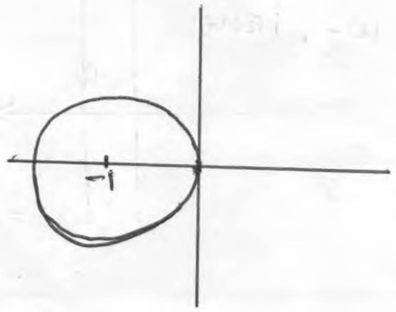
$$\Rightarrow \frac{(u^2+v^2) + 2u(u^2+v^2)}{(u^2+v^2)^2} = 0$$

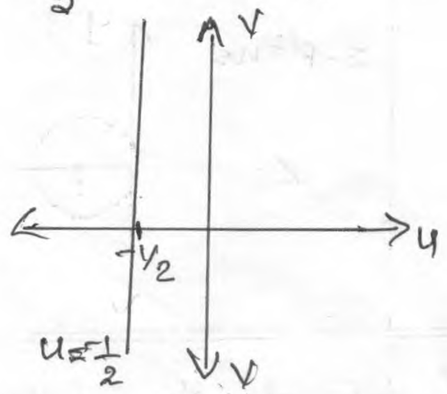
$$\Rightarrow (u^2+v^2) + 2u(u^2+v^2) = 0$$

$$\Rightarrow (u^2+v^2)\left[1+2u\right] = 0$$

$$\Rightarrow u^2+v^2 \neq 0$$

$\Rightarrow 1+2u = 0 \Rightarrow 2u = -1 \Rightarrow u = -\frac{1}{2}$ is straight line.





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5) Find the image of $|z-1|=1$ under the mapping $w = \frac{1}{z}$.

Ans: $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$\Rightarrow x+iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

Equating real and imaginary parts.

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

To find image of $|z-1|=1$,

$$|x+iy-1| = 1$$

$$\Rightarrow |(x-1)+i(y)| = 1$$

$$\sqrt{(x-1)^2+y^2} = 1$$

$$\Rightarrow (x-1)^2+y^2 = 1 \Rightarrow$$

$\Rightarrow$ circle with centre $(+1, 0)$ and radius 1.

$$\therefore (x-1)^2+y^2=1 \Rightarrow \left[\frac{u}{u^2+v^2} - 1\right]^2 + \left[\frac{-v}{u^2+v^2}\right]^2 = 1$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - \frac{2u}{u^2+v^2} = 0$$

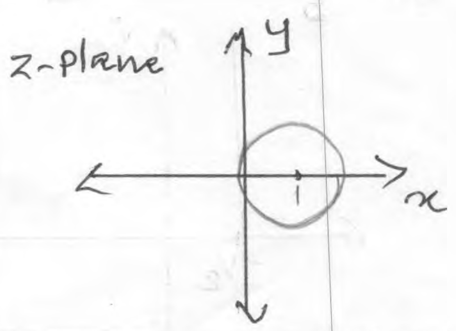
$$\frac{u^2+v^2}{(u^2+v^2)^2} - \frac{2u}{(u^2+v^2)} = 0$$

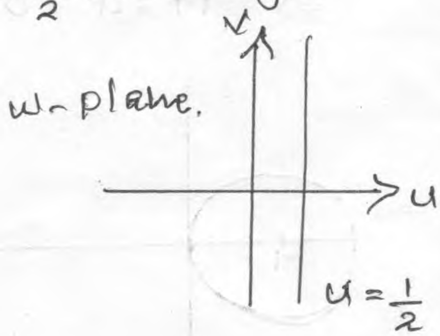
$$\Rightarrow (u^2+v^2) - 2u(u^2+v^2) = 0$$

$$\Rightarrow (u^2+v^2)[1-2u] = 0$$

since $u^2+v^2 \neq 0$, $1-2u = 0$

$\Rightarrow 2u = 1 \Rightarrow u = \frac{1}{2}$. straight line.





6. Find the images of the $1 < x < 2$ under the mapping $w = \frac{1}{z}$.

Ans: $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$x + iy = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

equating real and imaginary parts,

$$x = \frac{u}{u^2+v^2}, \qquad y = \frac{-v}{u^2+v^2}$$

to find images of $1 < x < 2$

consider $x > 1$, then $\frac{u}{u^2+v^2} > 1 \Rightarrow u > u^2+v^2$

$$\Rightarrow u^2+v^2-u < 0$$

This is the interior of the circle, $u^2+v^2-u+\frac{1}{4}-\frac{1}{4} = 0$

$$\Rightarrow u^2-u+\frac{1}{4}+v^2 = \frac{1}{4}$$

$$\Rightarrow u^2-u+\left(\frac{1}{2}\right)^2+v^2 = \frac{1}{4}$$

$$\Rightarrow \left(u-\frac{1}{2}\right)^2 + v^2 = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

with centre $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$

Now consider $x < 2$,

2) Find the image of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.

Ans:

$$w = \frac{1}{z}$$

$$z = \frac{1}{w} \Rightarrow x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

For $\frac{1}{4} \le y \le \frac{1}{3}$,

If $y = \frac{1}{4}$, then $\frac{1}{4} = \frac{-v}{u^2+v^2}$

$$\Rightarrow u^2+v^2 = -4v$$

$$u^2+v^2+4v = 0$$

$$u^2+[v^2+4v+4]-4 = 0$$

$$u^2+(v+2)^2 = 4 \Rightarrow (u-0)^2+(v-{}^{-}2)^2 = 4$$

$\Rightarrow$ Centre with $(0,-2)$ and radius 2.

If $y = \frac{1}{2}$,

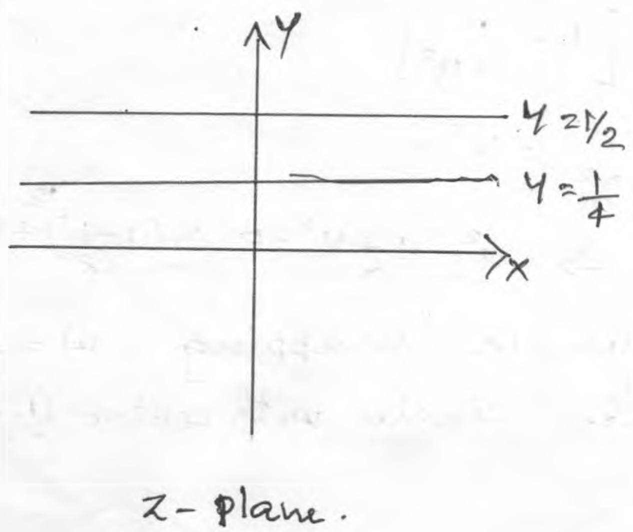
$$\frac{1}{2} = \frac{-v}{u^2+v^2}$$

$$\Rightarrow u^2+v^2 = -2v$$

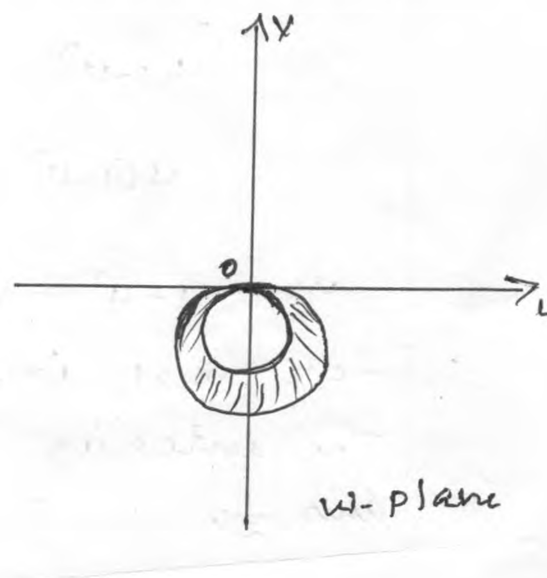
$$u^2+v^2+2v = 0$$

$$\Rightarrow u^2+(v+1)^2-1 = 0$$

$\Rightarrow u^2+(v+1)^2 = 1 \Rightarrow$ circle with $(0,-1)$ & radius 1.



z-plane.



w-plane

~~MODULE – II~~

~~CONFORMAL MAPPING~~

3b +6

~~LINEAR FRACTIONAL TRANSFORMATIONS [MOBIUS TRANSFORMATION]~~

~~$w = \frac{az+b}{cz+d}$~~

Q) Find the image of the region $x>1$ ($Re(z)>1$) under the mapping $w = \frac{1}{z}$

Soln:

$z = x+iy$ and $w = u+iv$

$$\therefore w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$w = \frac{x-iy}{x^2+y^2} \quad \text{——— ①}$$

Equating real and imaginary parts,

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

Using ① ⇒ on the line $x=1$, we've,

$$w = u+iv = \frac{1-iy}{1^2+y^2}$$

Thus $u = \frac{1}{1^2+y^2}$, $v = \frac{-y}{1+y^2}$

$$\therefore u(1-u) = \frac{1}{1+y^2}\left[1 - \frac{1}{1+y^2}\right]$$

$$u(1-u) = \frac{y^2}{(1+y^2)^2} = v^2$$

ie $u - u^2 - v^2 = 0 \Rightarrow u^2 - u + v^2 = 0 \Rightarrow \left(u-\frac{1}{2}\right)^2 + v^2 = \left(\frac{1}{2}\right)^2$

for $x>1$, image under mapping $w = \frac{1}{z}$ is the interior of the circle with centre $(\frac{1}{2}, 0)$ of radius $\frac{1}{2}$.

Eliminating α,

$$\sin^2\alpha + \cos^2\alpha = \frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k}$$

$\Rightarrow \dfrac{u^2}{\cosh^2 k} + \dfrac{v^2}{\sinh^2 k} = 1$ which is an ellipse in the w-plane with focci $(\pm 1, 0)$.

Fixed points or Invariant points.

Fixed points of a mapping $w = f(z)$ are points that are mapped on to themselves. The fixed-points are obtained by putting $w = z$.

Q) Find the fixed points of the mapping

$$w = \frac{3z - 5i}{iz - 1}$$

Soln: put $w = z$

$$\Rightarrow z = \frac{3z-5i}{iz-1}$$

$$iz^2 - z = 3z - 5i$$

$$iz^2 - 4z + 5i = 0$$

$$\Rightarrow z = \frac{4 \pm \sqrt{16 - 20i^2}}{2i} = \frac{4 \pm \sqrt{36}}{2i} = \frac{4 \pm 6}{2i}$$

$$\Rightarrow \frac{10}{2i}, \frac{-2}{2i}$$

$$\Rightarrow x^2+y^2-4y=0$$

Substituting for x^2 and y^2, we get

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2}+\frac{(-v)^2}{(u^2+v^2)^2}-4\frac{-v}{u^2+v^2}=0$$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2}+\frac{4v}{(u^2+v^2)}=0$$

$$\Rightarrow (u^2+v^2)+4v(u^2+v^2)=0$$

$$\Rightarrow (u^2+v^2)(1+4v)=0$$

$\Rightarrow 1+4v=0$, which is straight line.

Thus the image of the circle $|z-2i|=2$ is the straight line $1+4v=0$.

Q). MAPPING OF $w = \sin z$

Soln:

$$w=\sin z \Rightarrow u+iv=\sin(x+iy)$$

$$\Rightarrow u+iv=\sin x \cos iy+\cos x \sin iy$$

$$\Rightarrow u+iv=\sin x \cosh y+i\cos x \sinh y$$

$$\Rightarrow u=\sin x \cosh y, \quad v=\cos x \sinh y.$$

$\cos iy=\cosh y$ $\sin iy=i\sinh y$
$\cos 0=1$ $\sin 0=0$

Case 1:

Consider the real axis $y=0$ in the z-plane,

$$u=\sin x \cos 0$$

$$v=\cos x \sin 0$$

$\Rightarrow -5i \;,\; i$

$\therefore$ The fixed points are $z = -5i$ and i.

H.W

1) Find the fixed points of the transformations

$$w = \frac{5-4z}{4z-2}$$

2) Find the image of the following region under the mapping $w = \frac{1}{z}$.

$|z-3| = 5$.

CONFORMAL MAPPING

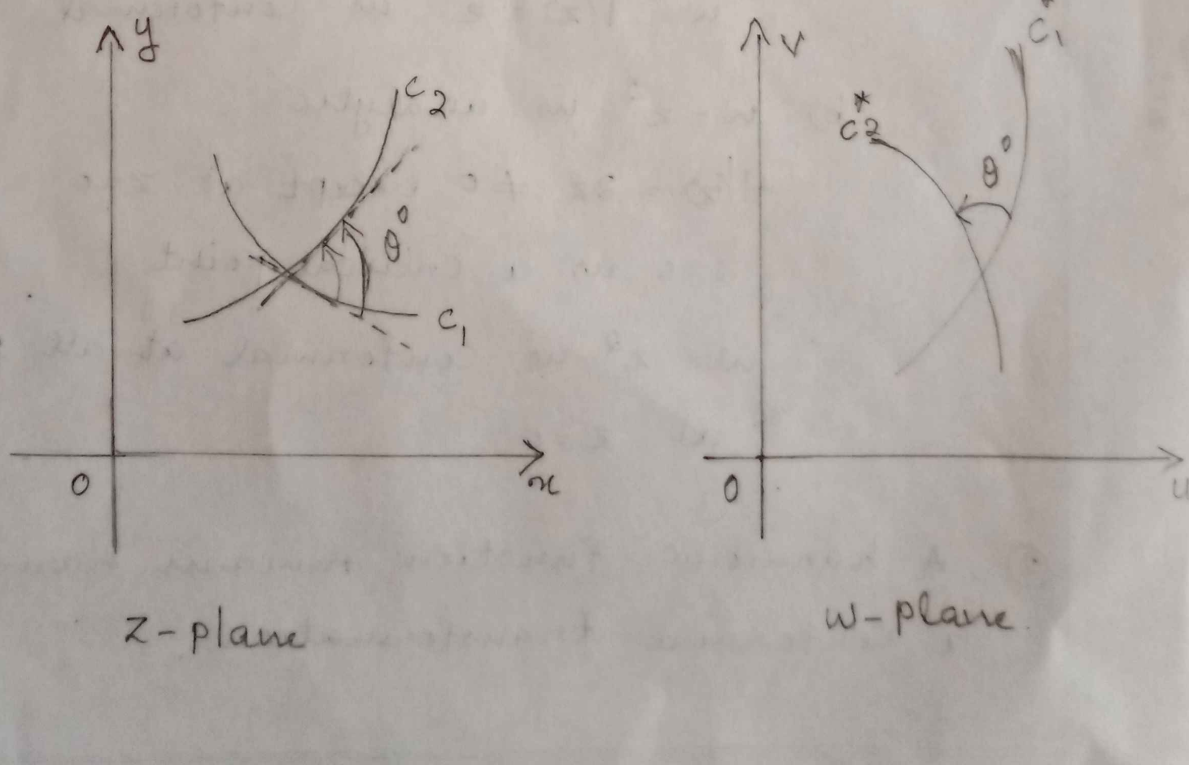
The complex function $w = f(z)$

ie $u+iv = f(x+iy)$

involves 4 variables, 2 independent variables x and y and 2 dependent variables u and v.

A 4-dimensional region is required to represent it graphically. As it is not possible, we choose two complex planes z-plane and w-plane.

In the z-plane we plot $z = x+iy$ and in the w-plane we plot $w = u+iv$. So the function $w = f(z)$ defines a correspondance between points of these two planes.



z-plane

w-plane

A transformation which preserves magnitude and sense of the angle between every pair of curves in some domain in the z-plane as its image in the w-plane is called Conformal mapping.

A transformation which preserves angles in magnitude alone is called isogonal mapping.

<u>Note</u>:

① If $w = f(z)$ is analytic then it is Conformal at each points of its domain, provided $f'(z) \neq 0$. The points at which $f'(z) = 0$ are called critical points.

Eg: a) $w = e^z$ is analytic

$f'(z) = e^z \neq 0$ for all z.

$\therefore w = f(z) = e^z$ is Conformal.

b) $w = z^2$ is analytic.

$f'(z) = 2z \neq 0$ except at $z = 0$

$z = 0$ is a critical point

$\therefore w = z^2$ is Conformal at all points except at $z = 0$.

② A harmonic function remains harmonic under a Conformal transformation.

Q) Discuss the mapping or transformation of $w = z^2$

Soln:

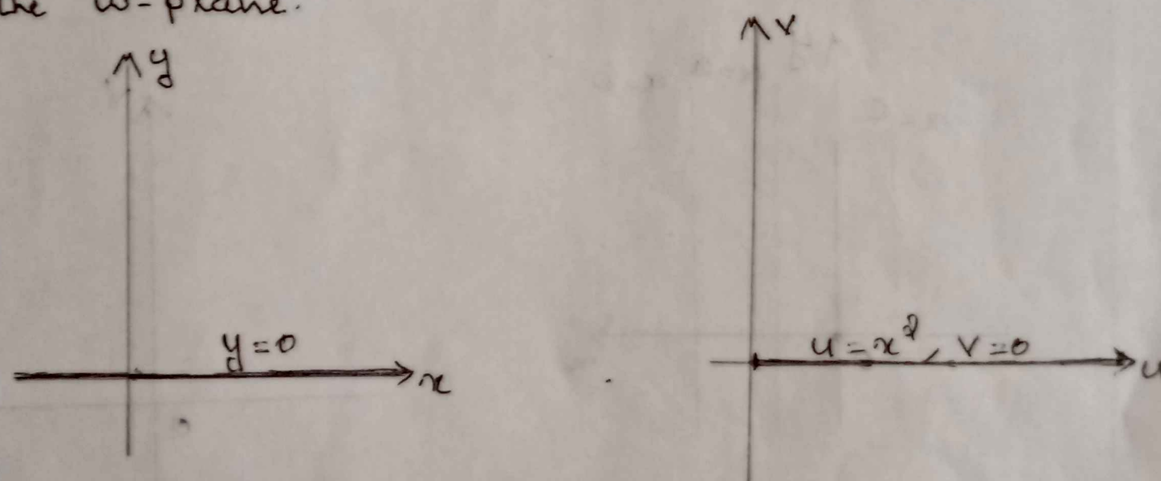
$$w = z^2$$

$$u+iv = (x+iy)^2 = x^2 - y^2 + i\,2xy.$$

$\Rightarrow u = x^2 - y^2$ and $v = 2xy$

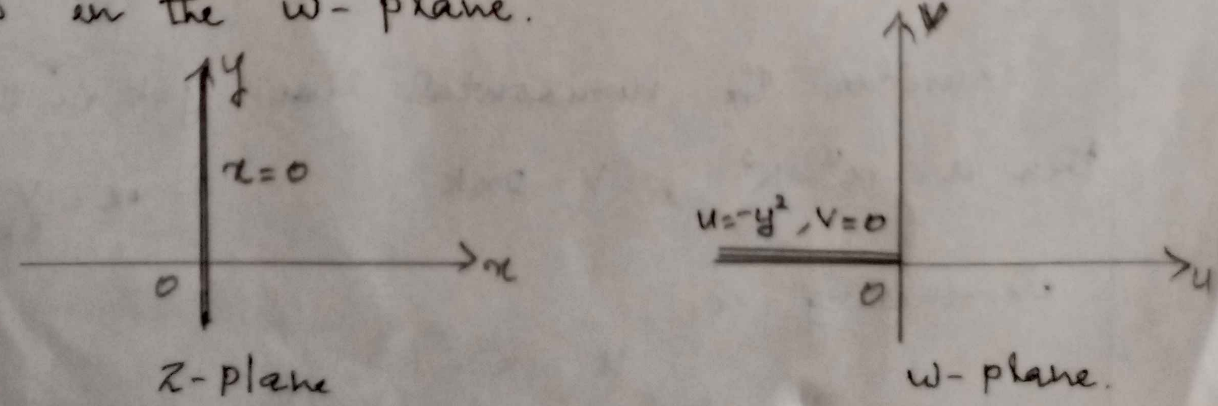
Case 1:

Consider the real axis $y=0$ in the z-plane. then $u = x^2$, $v = 0$; which is the positive real axis in the w-plane.



Case 2:-

Consider the imaginary axis $x=0$ in the z-plane then $u = -y^2$, $v=0$; which is the negative real axis in the w-plane.



Case 3:- Consider the vertical line $x = c$ in the z-plane then $u = c^2 - y^2$; $v = 2cy$.

eliminating y, $u = c^2 - y^2$

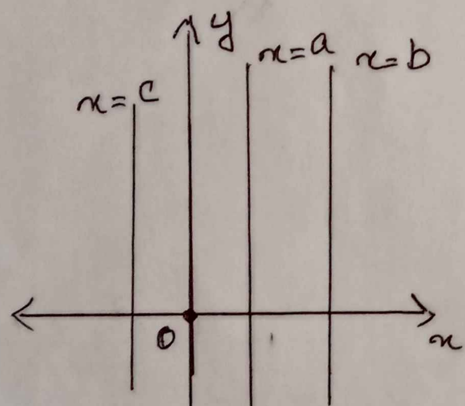
$$u = c^2 - \left(\frac{v}{2c}\right)^2 \qquad \therefore y = \frac{v}{2c}$$

$$\frac{v^2}{4c^2} = c^2 - u$$

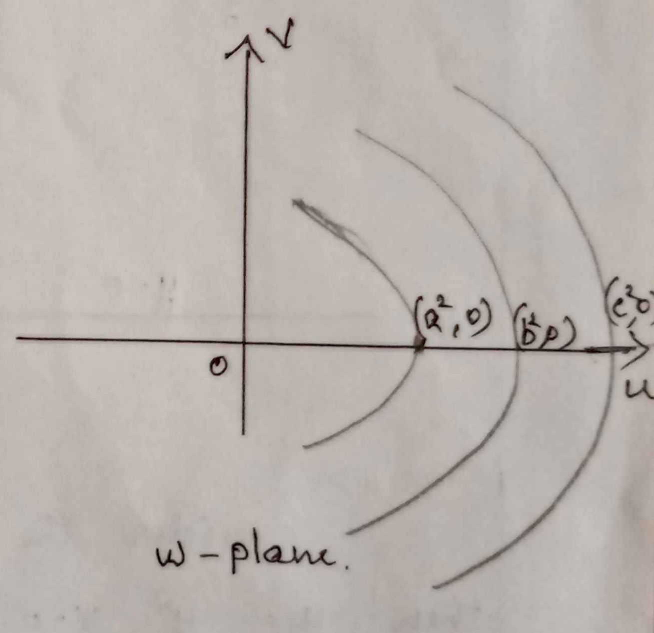
$$v^2 = 4c^2(c^2 - u)$$

$$(v-0)^2 = -4c^2(u - c^2),$$ which is a left handed parabola with vertex at $(x_1, y_1) = (c^2, 0)$.

> $(y - y_1)^2 = 4a(x - x_1)$
> Vertex at $(x_1, y_1) \Rightarrow$ parabola



Z-plane.



w-plane.

Case 4:

Consider the horizontal line $y = k$ in the z-plane. then $u = x^2 - k^2$, $v = 2xk$ $\qquad \therefore x = \frac{v}{2k}$

eliminating x,

$$u = x^2 - k^2$$

$$\Rightarrow u = x^2 - \left(\frac{v}{2k}\right)^2$$

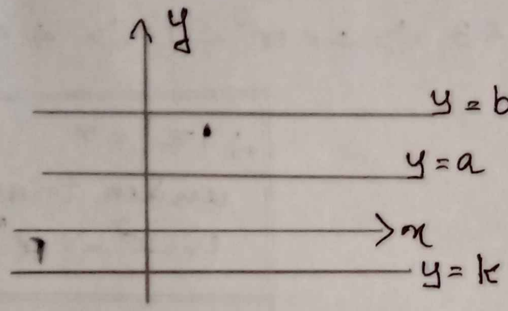
$$\Rightarrow \quad u = \left(\frac{v}{2k}\right)^2 - k^2$$

$$\frac{v^2}{4k^2} = u + k^2$$

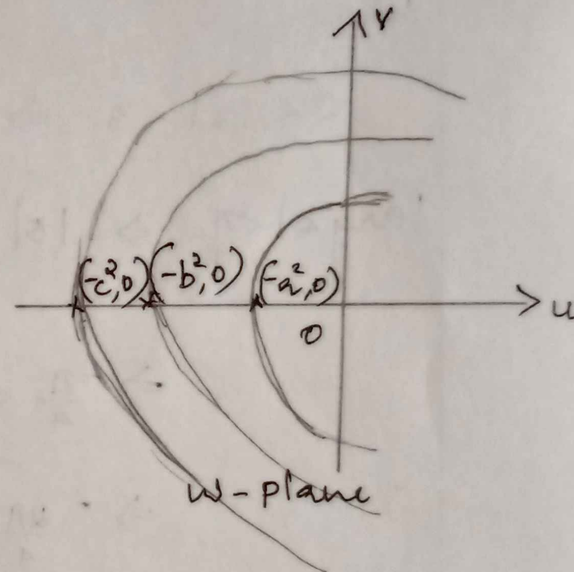
$$v^2 = 4k^2(u + k^2)$$

$$(v-0)^2 = 4k^2(u - -k^2)$$

which is a right handed parabola with vertex at $(-k^2, 0)$



z-plane



w-plane

Also $f'(z) = 2z \neq 0$ except at $z = 0$

Hence the transformation $w = z^2$ is conformal at all points except at $z = 0$.

REMARK

Using polar forms $z = re^{i\theta}$ in the z-plane, $w = Re^{i\phi}$ in the w-plane.

$$w = z^2 \Rightarrow Re^{i\phi} = (re^{i\theta})^2$$

$$\Rightarrow Re^{i\phi} = r^2 e^{i2\theta}$$

$$\Rightarrow R = r^2 \text{ and } \phi = 2\theta.$$

Hence the circle having radius r_0 is mapped to circle having radius r_0^2 and θ_0 mapped to $2\theta_0$

Q) Find the image of the region $2 < |z| < 3$; $|\arg z| < \frac{\pi}{4}$ under the map $w = z^2$.

Soln:-

$2 < |z| < 3 \Rightarrow 2 < r < 3 \Rightarrow 4 < r^2 < 9 \Rightarrow 4 < R < 9$

$|\arg z| < \frac{\pi}{4} \Rightarrow |\theta| < \frac{\pi}{4}$

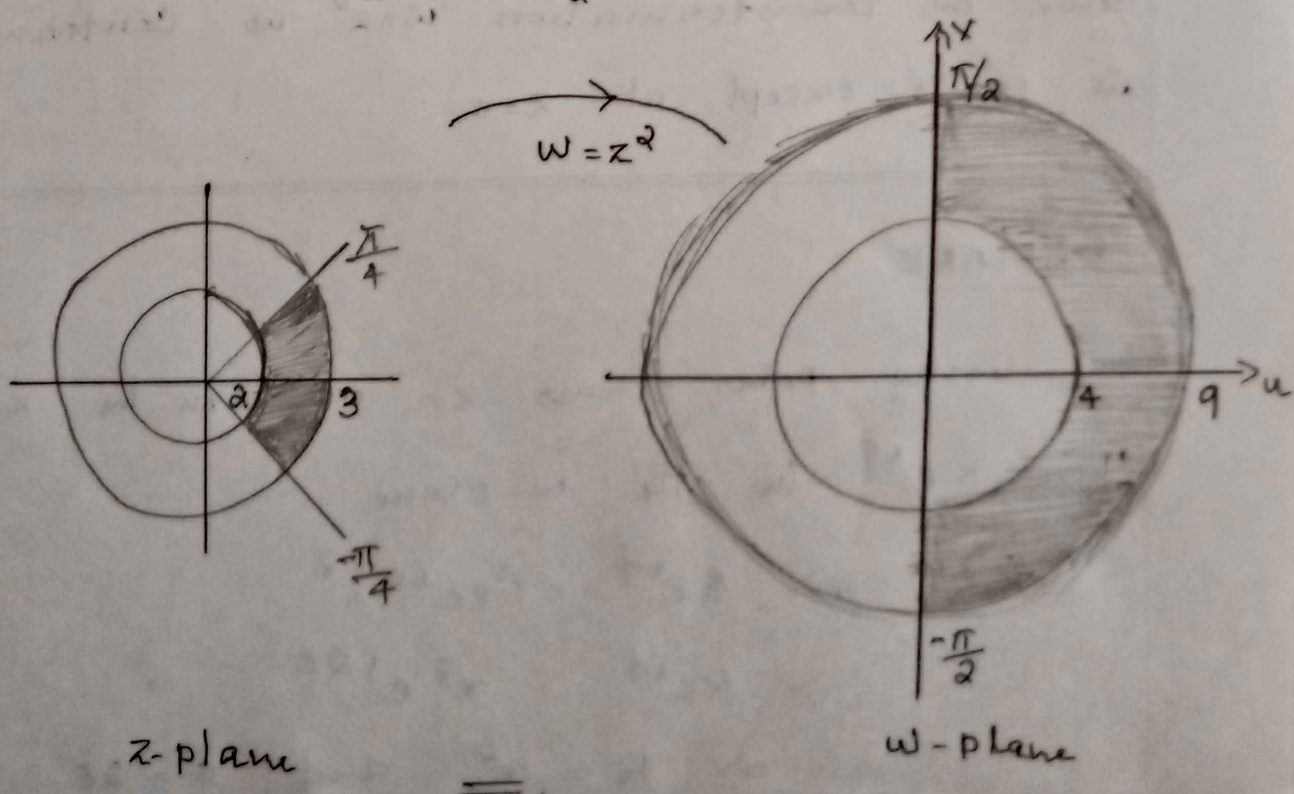
$\Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4}$

$\Rightarrow -\frac{2\pi}{4} < 2\theta < \frac{2\pi}{4}$

$\Rightarrow -\frac{\pi}{2} < \phi < \frac{\pi}{2}$

∴ $|z| = r$ under transformation $w = z^2 \Rightarrow R = r^2$ & $\phi = 2\theta$

∴ $\arg z = \theta$ under $w = z^2$, $\phi = 2\theta$.



MAPPING OF $w = e^z$

Q. Conformal mapping of $w = e^z$.

Soln:

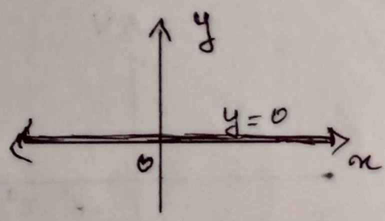
$$w = e^z \Rightarrow u + iv = e^{x+iy} = e^x . e^{iy}$$

$$\Rightarrow u + iv = e^x [\cos y + i \sin y]$$

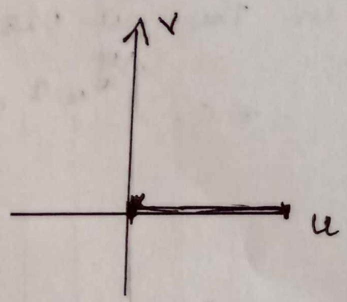
$$\Rightarrow u = e^x \cos y, \quad v = e^x \sin y.$$

Case 1:

Consider the real axis $y = 0$ in the z-plane then $u = e^x$; $v = 0$ which is positive real axis in the w-plane.



z-plane



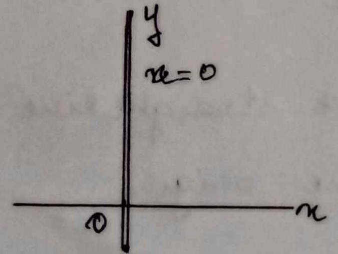
w-plane.

Case 2 :-

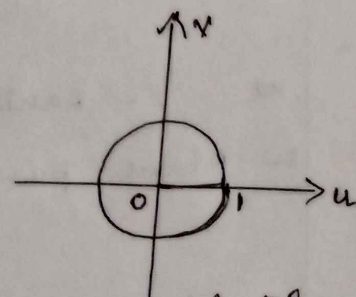
Consider the imaginary axis $x = 0$ in the z-plane then $u = \cos y$; $v = \sin y$.

Eliminating y, $\cos^2 y + \sin^2 y = u^2 + v^2$

ie $u^2 + v^2 = 1$, which is a circle in the w-plane with centre at $(0, 0)$ & radius 1.



z-plane



w-plane

Case 3:

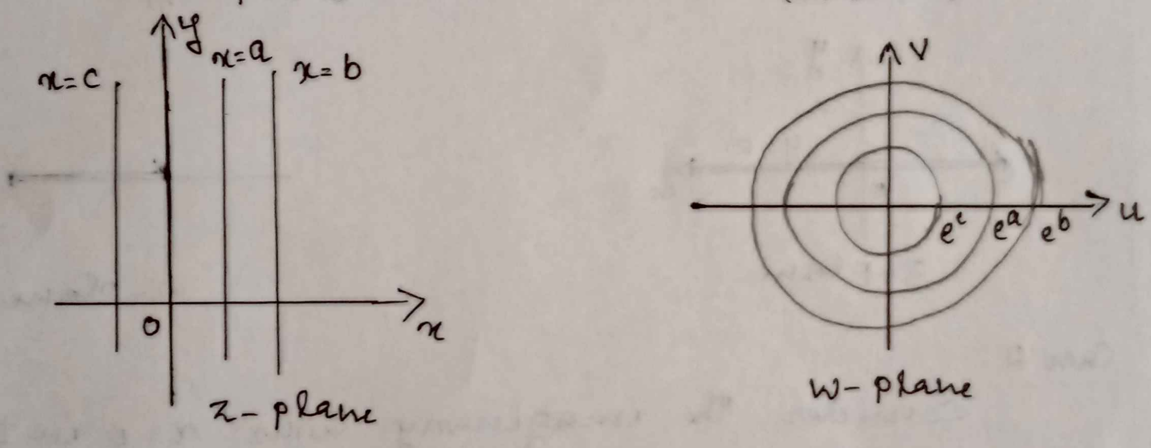
Consider the vertical line $x = c$ in the z-plane, then $u = e^c \cos y$; $v = e^c \sin y$

Eliminating y,

$$\cos^2 y + \sin^2 y = \left(\frac{u}{e^c}\right)^2 + \left(\frac{v}{e^c}\right)^2$$

ie $$1 = \frac{u^2}{(e^c)^2} + \frac{v^2}{(e^c)^2}$$

ie $u^2 + v^2 = (e^c)^2$ which is a circle in the w-plane with centre at (0, 0) & radius e^c.



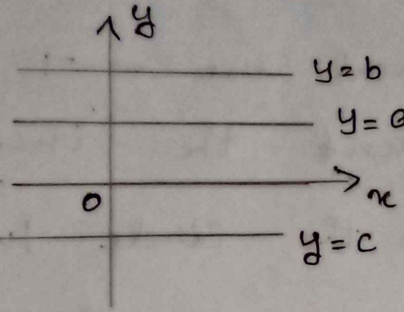
Case 4:-

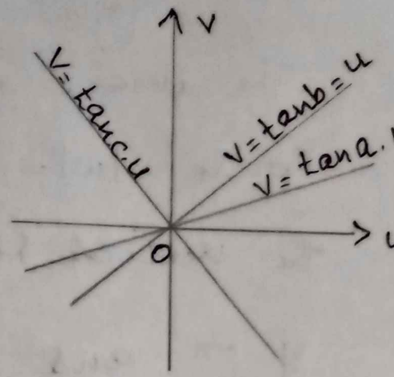
Consider the horizontal lines $y = k$ in the z-plane then $u = e^x \cos k$; $v = e^x \sin k$

Eliminating x,

$$\frac{v}{u} = \tan k$$

ie $v = \tan k \cdot u$; which is a straight line in the w-plane passing through the origin.





Q) Find the image of the region $-\log 2 \le x \le \log 4$ under the mapping $w = e^z$.

Soln:

$$w = e^z$$

$$-\log 2 \le x \le \log 4$$

Image of the vertical line $x = -\ln 2$ is the circle

$$|w| = e^{-\ln 2}$$

$$\Rightarrow |w| = e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}$$

Similarly, the image of the line $x = \ln 4$ is the circle $|w| = e^{\ln 4} \Rightarrow |w| = 4$.

Hence the region bounded by the lines $|w| = \frac{1}{2}$ and $|w| = 4$.

Q) Find the image of the region $-1 \le x \le 2$, $-\pi \le y \le \pi$ under the mapping $w = e^z$

Soln:

$-1 \le x \le 2 \Rightarrow x = -1, \ x = 2$

$-\pi \le y \le \pi \Rightarrow y = -\pi, \ y = \pi$.

The image of the vertical line $x = -1$ is the Circle $|w| = e^{-1}$ ~~is the~~ and the image of the vertical line $x = 2$ is the Circle $|w| = e^{2}$.

$y = -\pi$ and $y = \pi$ are mapped on the rays $\arg w = -\pi$ and $\arg w = \pi$ respectively.

Q) <u>Mapping of</u> $w = \frac{1}{z}$

<u>Soln</u>:-

$$u + iv = \frac{1}{x+iy}$$

$$\Rightarrow u + iv = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{-y}{x^2+y^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = \frac{-v}{u^2+v^2}$$

Q) Find the image of the following regions under the mapping $w = \frac{1}{z}$.

1) $\frac{1}{4} < y < \frac{1}{2}$

2) $0 < y < \frac{1}{2}$

3) $|z - 2i| = 2$.

Soln:

(1) $\frac{1}{4} < y < \frac{1}{2}$.

$\frac{1}{4} < y$ and $y < \frac{1}{2}$.

Let $w = \frac{1}{z} = u + iv$

Then $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$

$\frac{1}{4} < y \Rightarrow \frac{1}{4} < \frac{-v}{u^2+v^2}$

$\Rightarrow u^2+v^2 < -4v$

$\Rightarrow u^2+v^2+4v < 0$ which is the interior part of the circle.

Now $y < \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2}$

$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2 > -2v$

$\Rightarrow u^2+v^2+2v > 0$ which is the exterior part of the circle.

$\therefore$ The image of the region $\frac{1}{4} < y < \frac{1}{2}$ is mapped in to the region between the circles $u^2+v^2+4v = 0$ and $u^2+v^2+2v = 0$.

(2) $0 < y < \frac{1}{2}$

Let $w = \frac{1}{z} = u + iv$

$\Rightarrow x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

$0<y \Rightarrow 0< \frac{-v}{u^2+v^2}$

$\Rightarrow -v>0 \Rightarrow v>0$ ∴ The image of the $0<y$ is $v>0$, which is the upper half plane

Now $y<\frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2}<\frac{1}{2}$

$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2+2v>0$, which is the exterior part of the circle.

∴ $0<y<\frac{1}{2}$ ⇒ Images of $0<y<\frac{1}{2}$ is mapped onto $v>0$ and $u^2+v^2+2v>0$

③ $|z-2i|=2$

Let $w=\frac{1}{z}=u+iv$

$\Rightarrow x=\frac{u}{u^2+v^2}$ and $y=\frac{-v}{u^2+v^2}$

Now $|z-2i|=2 \Rightarrow |x+iy-2i|=2$

$\Rightarrow |x+i(y-2)|=2$

$\Rightarrow \sqrt{x^2+(y-2)^2}=2$ ∴ $|x+iy|=\sqrt{x^2+y^2}$

Squaring both sides,

$\Rightarrow x^2+(y-2)^2=4$

$\Rightarrow x^2+y^2-4y+4=4$

$\Rightarrow x^2+y^2-4y=0$

Substituting for x^2 and y^2, we get

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}-4\frac{-v}{u^2+v^2}=0$$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2}+\frac{4v}{(u^2+v^2)}=0$$

$$\Rightarrow u^2+v^2+4v(u^2+v^2)=0$$

$$\Rightarrow (u^2+v^2)(1+4v)=0$$

$\Rightarrow 1+4v=0$, which is straight line.

Thus the image of the circle $|z-2i|=2$ is the straight line $1+4v=0$.

Q). MAPPING OF $w = \sin z$

Soln:

$w=\sin z \Rightarrow u+iv=\sin(x+iy)$

$\Rightarrow u+iv=\sin x\cos iy+\cos x\sin iy$

$\Rightarrow u+iv=\sin x\cosh y+i\cos x\sinh y$

$\Rightarrow u=\sin x\cosh y$, $v=\cos x\sinh y$.

$\cos iy=\cosh y$ $\sin iy=i\sinh y$
$\cos 0=1$ $\sin 0=0$

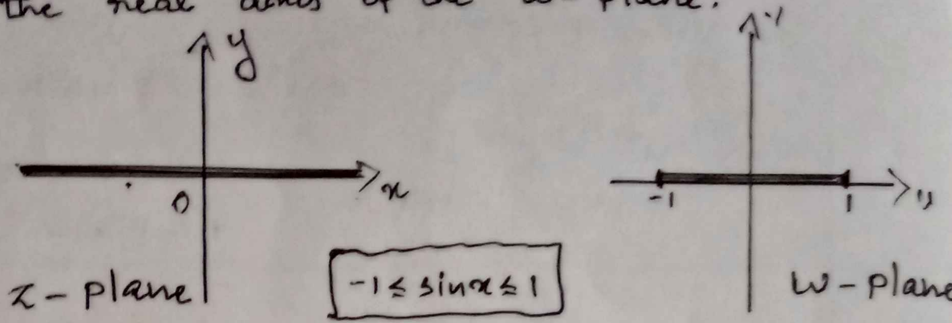
Case I:

Consider the real axis $y=0$ in the z-plane,

$u=\sin x\cos 0$

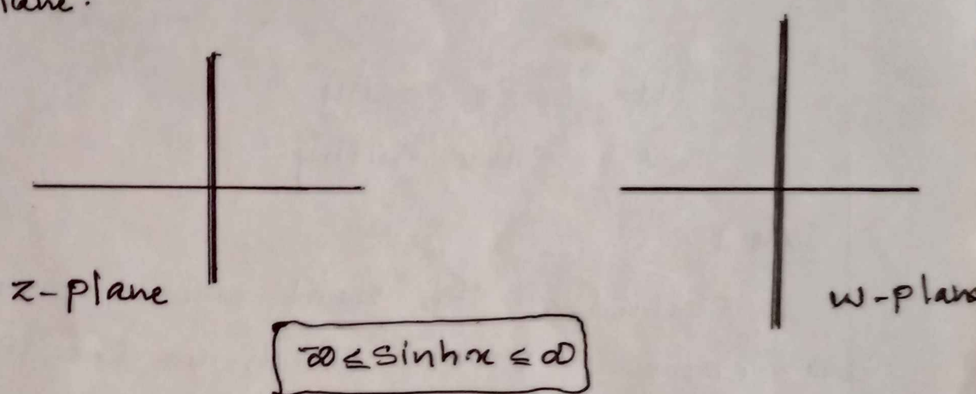
$v=\cos x\sin 0$

$u = \sin x$, $v = 0$ which is the line segment from -1 to 1 on the real axis of the w-plane.



Case II

consider the imaginary axis $x = 0$ in the z-plane, then $u = 0$; $v = \sinh y$ which is the imaginary axis of the w-plane.



Case III

consider the vertical line $x = c$ in the z-plane, then

$u = \sin c \cosh y$; $v = \cos c \sinh y$

eliminating y,

$$\cosh^2 y - \sinh^2 y = \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c}$$

ie $$1 = \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c}.$$ since $\cosh^2\theta - \sinh^2\theta = 1$

$$\Rightarrow \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$$; which is a hyperbola with foci $(\pm 1, 0)$.

Case IV

consider the horizontal line $y = k$ in the z-plane, then $u = \sin x \cosh k$, $v = \cos x \sinh k$

Eliminating α,

$$\sin^2\alpha + \cos^2\alpha = \frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k}$$

$\Rightarrow \frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$ which is an ellipse in the w-plane with focci $(\pm 1, 0)$.

Fixed points or Invariant points.

Fixed points of a mapping $w = f(z)$ are points that are mapped on to themselves. The fixed-points are obtained by putting $w = z$.

Q) Find the fixed points of the mapping

$$w = \frac{3z - 5i}{iz - 1}.$$

Soln:

put $w = z$

$$\Rightarrow z = \frac{3z - 5i}{iz - 1}$$

$$iz^2 - z = 3z - 5i$$

$$iz^2 - 4z + 5i = 0$$

$$\Rightarrow z = \frac{4 \pm \sqrt{16 - 20i^2}}{2i} = \frac{4 \pm \sqrt{36}}{2i} = \frac{4 \pm 6}{2i}$$

$$\Rightarrow \frac{10}{2i}, \frac{-2}{2i}$$

$\Rightarrow -5i \ , \ i$

$\therefore$ The fixed points are $z = -5i$ and i.

H.W

1) Find the fixed points of the transformations

$$w = \frac{5-4z}{4z-2}$$

2) Find the image of the following region under the mapping $w = \frac{1}{z}$.

$|z-3| = 5$.

MODULE - IV

COMPLEX INTEGRATION

* Complex Line Integral

Let $f(z)$ be a continous function of z, defined at all points of a curve c having end points A and B, then the complex integral of $f(z)$ along c is $\int_c f(z)\,dz$.

'C' is called the path of integration. If c is a closed curve the integral is called contour integral and is denoted by $\oint_c f(z)\,dz$.

* FIRST EVALUATION METHOD

Let $f(z)$ be analytic in a simple connected domain D and $F(z)$ be an analytic function such that $F'(z) = f(z)$. then,

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

Eg: $\displaystyle\int_0^{1+i} z^2\,dz = \left.\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3}$

$$= \frac{1+3i-3-i}{3} = \frac{-2+2i}{3}$$

* SECOND EVALUATION METHOD

The first evaluation method is restricted to analytic functions. But the second evaluation method can be applied to any continous complex functions.

$$z = x+iy$$
$$dz = dx + i\,dy$$

ie $$\int_C f(z)\,dz = \int_{t=a}^{b} f(z(t))\,\frac{dz}{dt}\,dt$$

$$= \int f(x+iy)(dx+i\,dy)$$

Q)1. Evaluate $\int_C \bar{z}\,dz$ where C is given by $x = 3t$, $y = t^2-1$, $-1 \le t \le 4$.

Soln:

$$\int_C \bar{z}\,dz = \int (x-iy)(dx+i\,dy) \quad \because dz = dx+i\,dy$$

$$= \int_{t=-1}^{4} [3t - i(t^2-1)]\,[3\,dt + i\,2t\,dt]$$

$$= \int_{-1}^{4} (3t - it^2 + i)(3+2it)\,dt$$

$$= \int_{-1}^{4} (9t + 6it^2 - 3it^2 + 2t^3 + 3i - 2t)\,dt$$

$$= \int_{-1}^{4} (7t + 3it^2 + 2t^3 + 3i)\,dt$$

$$= \left[\frac{7t^2}{2} + 3i\frac{t^3}{3} + 2\frac{t^4}{4} + 3it\right]_{-1}^{4}$$

$$= (56 + 64i + 128 + 12i) - \left(7/2 - i + \frac{1}{2} - 3i\right)$$

$$= 180 + 80i$$

2\. Evaluate $\int_0^{1+2i} f(z)\,dz$ where $f(z) = Re(z)$

i) Along the straight line from 0 to $1+2i$

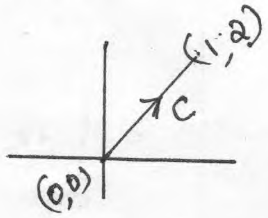
ii) Along the real axis from $z=0$ to $z=1$ and then along a line parallel to imaginary axis from $z=1$ to $z=1+2i$

Soln:

$f(z) = Re(z) = x.$

i) Along the straight line from 0 to $1+2i$

$\int_C f(z)\,dz = \int_C x\,(dx + i\,dy)$



Along C; $y = 2x$

$\Rightarrow dy = 2\,dx;$

$\Rightarrow x$ from 0 to 1

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$(x_1, y_1) = (0,0)$

$(x_2, y_2) = (1,2)$

$\therefore m = \frac{2-0}{1-0} = 2$

$\therefore \int_C f(z)\,dz = \int_{x=0}^{1} x\,(dx + i\,2\,dx)$

$= (1+2i)\int_0^1 x\,dx$

$= (1+2i)\left[\frac{x^2}{2}\right]_0^1$

$= (1+2i)\,\frac{1}{2} = \frac{1}{2} + i$

※ $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$\Rightarrow \int_C f(z)\,dz = \frac{1}{2} + i$

ii) Along the real axis from $z=0$ to $z=1$ and parallel to Imaginary axis from $z=1$ to $z=1+2i$

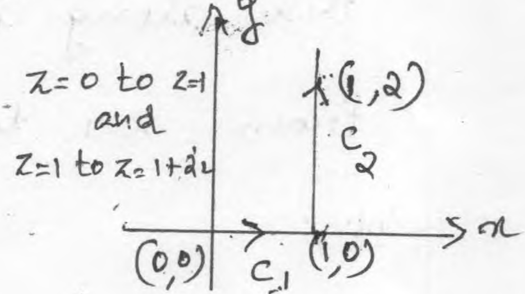
Soln: $z = 0 \Rightarrow (0,0)$ $\quad z = x+iy$

$z = 1 \Rightarrow (1,0)$

and parallel to imaginary axis from $z=1$ to $z=1+2i$

$\Rightarrow z=1 \Rightarrow (1,0)$

$z=1+2i \Rightarrow (1,2)$



$C_1: \Rightarrow (0,0)$ to $(1,0)$

$C_2: \Rightarrow (1,0)$ to $(1,2)$

$\therefore C = C_1 + C_2$.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$= \int_{C_1} x\,(dx+i\,dy) + \int_{C_2} x\,(dx+i\,dy)$$

Along C_1, $y = mx = \frac{0-0}{1-0}x = 0 \Rightarrow y=0$.

$\Rightarrow dy = 0$; x from 0 to 1.

Along C_2, $(1,0)$ to $(1,2)$

$\Rightarrow x=1 \Rightarrow dx = 0$

$\Rightarrow y$; 0 to 2.

$$\therefore \int_C f(z)\,dz = \int_{x=0}^{1} x\,dx + \int_{y=0}^{2} 1 \cdot i\,dy.$$

$$= \left[\frac{x^2}{2}\right]_0^1 + i\left[y\right]_0^2$$

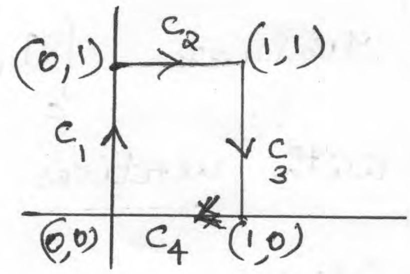
$$= \frac{1}{2} + 2i$$

Q) Evaluate $\oint_c Re(z^2)\,dz$ over the boundary c of the square with vertices 0, i, 1+i, 1 clockwise.

Soln:

$z^2 = (x^2 - y^2) + i\,2xy$

$Re(z^2) = x^2 - y^2$



Along c_1, $x = 0 \Rightarrow dx = 0$; y : 0 to 1.

Along c_2, $y = 1 \Rightarrow dy = 0$; x : 0 to 1

Along c_3, $x = 1 \Rightarrow dx = 0$; y : 1 to 0

Along c_4, $y = 0 \Rightarrow dy = 0$; x : 1 to 0.

$$\therefore \int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + \int_{c_4} f(z)\,dz$$

$$\Rightarrow \int_{c_1}(x^2-y^2)(dx + i\,dy) + \int_{c_2}(x^2-y^2)(dx+i\,dy) + \int_{c_3}(x^2-y^2)(dx+i\,dy) + \int_{c_4}(x^2-y^2)(dx+i\,dy)$$

$$= \int_{y=0}^{1} -y^2\, i\,dy + \int_{x=0}^{1}(x^2-1)\,dx + \int_{y=1}^{0}(1-y^2)\,i\,dy + \int_{x=1}^{0} x^2\,dx$$

$$= i\left[\frac{-y^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - x\right]_0^1 + i\left[y - \frac{y^3}{3}\right]_1^0 + \left[\frac{x^3}{3}\right]_1^0$$

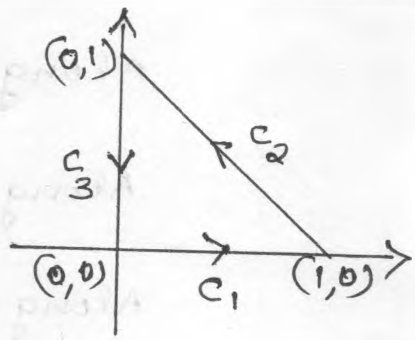
$$\Rightarrow \left(\frac{-i}{3}\right) + \left(\frac{1}{3} - 1\right) + i\left(\frac{-2}{3}\right) + \frac{-1}{3}$$

$$\Rightarrow \quad = \quad \frac{-i}{3} - \frac{2}{3} - \frac{2}{3}i - \frac{1}{3} = -1-i$$

$$\therefore \oint_c f(z)\, dz = -1-i$$

Q) Evaluate $\int_c Im(z^2)\, dz$ where c is the triangle with vertices $0, 1, i$ counter clock wise.

Soln:



$$z^2 = x^2 - y^2 + 2ixy$$

$$\therefore Im(z^2) = 2xy.$$

Along c_1, $y=0$, $dy=0$, x; 0 to 1.

Along c_2, $y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$\Rightarrow y = -x+1 \Rightarrow dy = -dx, \; x: 1 \text{ to } 0$$

Along c_3, $x=0 \Rightarrow dx = 0$, $y: 1$ to 0.

$$\therefore \int_c Im(z^2)\, dz = \int_{c_1} 2xy\,(dx + i dy) + \int_{c_2} 2xy\,(dx+idy) + \int_{c_3} 2xy\,(dx + i dy)$$

$$\Rightarrow \quad = \int_{x=0}^{1} 0 + \int_{x=1}^{0} 2x(-x+1)(dx - i dx) + \int_{y=1}^{0} 0.$$

$$= \int_{x=1}^{0} 2x(-x+1)(1-i)\,dx$$

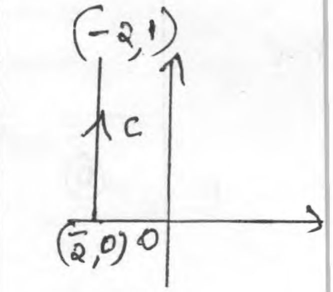
$$= (1-i)\int_{x=1}^{0} (-2x^2 + 2x)\,dx$$

$$= (1-i)\left[-\frac{2x^3}{3} + \frac{2x^2}{2}\right]_1^0$$

$$= (1-i)\left[\frac{2}{3} - 1\right] = (1-i)\frac{-1}{3}$$

$$= \frac{-(1-i)}{3}$$

Q) Show that $\int_C (2+z)^2\,dz = \frac{-i}{3}$ where C is any path connecting the points -2 & $-2+i$.



-2 to $-2+i$

$\Rightarrow (-2,0)$ to $(-2,1)$

Soln:

$$\int_C (2+z)^2\,dz = \int_C (2+x+iy)^2(dx+i\,dy)$$

Along C, $x = -2 \Rightarrow dx = 0$; $y = 0$ to 1

$$\therefore \int_C (2+z)^2\,dz = \int_{y=0}^{1} (2-2+iy)^2 (i\,dy)$$

$$= \int_0^1 (iy)^2\, i\,dy$$

$$= -i\int_0^1 y^2\,dy$$

$$= -i\left[\frac{y^3}{3}\right]_0^1 = \frac{-i}{3}$$

$$\therefore \int_C (2+z)^2\,dz = \frac{-i}{3} \;//.$$

*

CAUCHY'S INTEGRAL THEOREM

If f(z) is analytic within or on a closed path c then $\oint_c f(z)\,dz = 0$

Eg:- ① $\oint_c \frac{1}{z^2+1}\,dz$; c : $|z| = \frac{1}{2}$

$z^2+1=0$
$z^2=-1$
$\Rightarrow z=\pm\sqrt{-1}$
$\Rightarrow z=\pm i$
$(z=i, -i)$

Here f(z) is not analytic at $z=\pm i$

at $z=i$, $|+i| = |0+1i| = 1 > \frac{1}{2}$

at $z=-i$, $|-i| = |0-1i| = 1 > \frac{1}{2}$

∴ The points $z=\pm i$ lies outside c.

∴ f(z) is analytic inside c.

By Cauchy's Integral theorem, $\oint_c f(z)\,dz = 0$

$$\Rightarrow \oint_c \frac{1}{z^2+1}\,dz = 0$$

② $\oint_c \frac{1}{z-3}\,dz$ C ; $|z|=1$

Here f(z) is not analytic at $z=3$

$|3| = 3 > 1$, lies outside c ; $|z|=1$.

∴ f(z) is analytic inside c.

∴ By cauchy's integral theorem, $\oint_c f(z)\,dz = 0$

$$\Rightarrow \oint_c \frac{1}{z-3}\,dz = 0$$

CAUCHY'S INTEGRAL FORMULA (CIF)

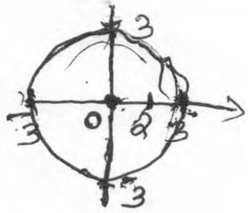
Let $f(z)$ be analytic within or on a closed contour c, then $\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$, z_0 is a point within c, the integration is being taken in the counter clockwise.

Also $$\oint_c \frac{f(z)}{(z-z_0)^2}\,dz = \frac{2\pi i}{1!} f'(z_0)$$

$$\oint_c \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i}{2!} f''(z_0)$$

In general, $$\oint_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0).$$

Q) Evaluate $\oint_c \frac{\cos\pi z}{z-2}\,dz$; $c : |z| = 3$



Soln:-

Here $\frac{\cos\pi z}{z-2}$ is not analytic at $z = 2$.

$|a+ib| = \sqrt{a^2+b^2}$

But when $z = 2$, $|z| = 2 < 3$

$\therefore$ $z = 2$ lies within c.

By Cauchy's Integral formula,

$z_0 = 2$

$$\oint_c \frac{\cos\pi z}{z-2}\,dz = 2\pi i\, f(2), \text{ where,}$$

$f(z) = \cos\pi z$

$f(2) = \cos 2\pi = 1$

$= 2\pi i \times 1 = \underline{2\pi i}$

Q) Evaluate $\oint_C \frac{e^z}{z+1} dz$ where C is

i) $|z| = 2$

ii) $|z| = \frac{1}{2}$

(iii) $|z+1| = \frac{1}{2}$.

Soln:

$f(z) = \frac{e^z}{z+1}$ is not analytic at $z = -1$.

i) $|z| = 2$

when $z = -1$, $|z| = |-1| = 1 < 2$.

$\therefore z = -1$ lies inside C.

$\therefore z_0 = -1$

By Cauchy's Integral formula (CIF),

$$\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i \times f(z_0)$$

$z_0 = -1$

$$\Rightarrow \oint_C \frac{e^z}{(z-(-1))} dz = 2\pi i \times f(-1), \quad \text{where } f(z) = e^z$$

$$= 2\pi i \times e^{-1} \qquad f(z_0) = f(-1) = e^{-1}$$

$$= \frac{2\pi i}{e}$$

---

ii) $|z| = \frac{1}{2}$

Soln: C is $|z| = \frac{1}{2}$

$z_0 = -1$.

when $z = -1$, $|z| = |-1| = 1 > \frac{1}{2}$

$\therefore z = -1$ lies Outside C.

ie ~~$z_0 = -1$~~, apply Cauchy's Integral thm.

$\therefore$ By Cauchy's Integral theorem,

$$\oint_c f(z)\, dz = 0$$

$$\Rightarrow \oint_c \frac{e^z}{z+1}\, dz = 0.$$

iii) $|z+1| = \frac{1}{2}$

when $z=-1$, $|z+1| = |0| = 0 < \frac{1}{2}$

$\therefore z=-1$ lies inside c.

$\therefore$ By CIF, $\oint_c \frac{e^z}{z+1}\, dz = 2\pi i \times f(-1)$

$= 2\pi i \times e^{-1} = \frac{2\pi i}{e}$

where $f(z) = e^z$

$f(z_0) = f(-1) = e^{-1}$

Q) Calculate $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z-2)}$; c is $|z| = 3$.

Soln:

Here $(z+1)(z-2) = 0$; c; $|z| = 3$

$\Rightarrow z = -1, 2$ are singular points.

when $z=-1$, $|z| = |-1| = 1 < 3 \Rightarrow z=-1$ lies inside c.

when $z = 2$, $|z| = |2| = 2 < 3 \Rightarrow z=2$ lies inside c.

$\therefore z=-1, 2$ lies inside c.

$\therefore z_0 = -1, 2$

CIF $\Rightarrow \oint_c \frac{f(z)}{z-z_0}\, dz = 2\pi i \times f(z_0)$

Let $\frac{1}{(z+1)(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z-2)}$

$\therefore 1 = A(z-2) + B(z+1)$

put $z=2 \Rightarrow 3B = 1 \Rightarrow$ $B = \frac{1}{3}$

put $z = -1 \Rightarrow -3A = 1 \Rightarrow \boxed{A = \frac{-1}{3}}$

$$\therefore \frac{1}{(z+1)(z-2)} = \frac{\frac{-1}{3}}{z+1} + \frac{1/3}{z-2}$$

$$\therefore \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z-2)} = \left(-1/3\right)\frac{\sin\pi z^2 + \cos\pi z^2}{z+1} + \left(1/3\right)\frac{\sin\pi z^2 + \cos\pi z^2}{z-2}.$$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z-2)} = \frac{-1}{3}\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z+1} + \frac{1}{3}\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}$$

$$\Rightarrow \quad = -\frac{1}{3} \times 2\pi i \times f(-1) + \frac{1}{3} 2\pi i \times f(2)$$

where $f(z) = \sin\pi z^2 + \cos\pi z^2$

$f(-1) = \sin\pi + \cos\pi$

$= 0 + -1$

$\Rightarrow f(-1) = \underline{\underline{-1}}$

$f(2) = \sin 4\pi + \cos 4\pi$

$= 0 + 1$

$\Rightarrow f(2) = 1$

$$\Rightarrow \quad = \frac{-1}{3} \times 2\pi i \times 1 + \frac{1}{3} \times 2\pi i \times 1$$

$$= \frac{2\pi i}{3} + \frac{2\pi i}{3}$$

$$= \underline{\underline{\frac{4}{3}\pi i}}$$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z+1)(z+2)} = \underline{\frac{4}{3}\pi i}$$

Q) Evaluate $\oint_c \frac{e^z}{(z+1)^3} dz$; c is $|z+1| = 1$

Soln:

$z = -1$ is a singular point.

when $z = -1$, ~~$|z| = |-1| = 1 <$~~ $|z+1| = |-1+1| = 0 < 1$

$\therefore z_0 = -1$ lies inside c.

$\therefore$ By CIF, $\oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$

$$\Rightarrow \oint_c \frac{e^z}{(z - -1)^3} dz = \oint_c \frac{e^z}{(z - -1)^{2+1}} dz$$

$$= \frac{2\pi i}{2!} \cdot f''(z_0)$$

$$= \frac{2\pi i}{2!} \times e^{-1}$$

$$\therefore \oint_c \frac{e^z}{(z+1)^3} dz = \frac{\pi i}{e}$$

where $f(z) = e^z$

~~$f(z_0) = f(-1) = e^{-1}$~~

$f'(z) = e^z$

$f''(z) = e^z$

$f''(z_0) = f''(-1) = e^{-1}$

Q) Evaluate $\oint \frac{\sin 2z}{z^4} dz$; c is $|z| = 1$

Soln:

$z = 0$ is a singular point.

when $z = 0$, $|z| = 0 < 1$

$\therefore z = z_0 = 0$ lies inside c.

$\therefore$ By CIF, $\oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} \times f^n(z_0)$.

ie $\oint_c \frac{\sin 2z}{z^4} dz = \oint_c \frac{\sin 2z}{(z-0)^{3+1}} dz$, $n=3$; $z_0=0$

$= \frac{2\pi i}{3!} f'''(z_0)$ where, $f(z) = \sin 2z$

$f'(z) = 2\cos 2z$

$f''(z) = -4\sin 2z$

$f'''(z) = -8\cos 2z$

$f'''(z_0) = f'''(0) = -8\cos 0$

$= -8$.

$= \frac{2\pi i}{3!} \times -8$

$= \frac{2\pi i}{6} \times -8$

$= \frac{-8}{3}\pi i$

Q) Calculate $\oint_c \frac{z^2+5z+3}{(z-2)^2}$; c is $|z|=3$

Soln:

$z=2$ is a singular point.

when $z=2$, $|z| = 2 < 3$

$\therefore z_0 = 2$ lies inside c.

By CIF, $\oint_c \frac{z^2+5z+3}{(z-2)^2} dz = \oint_c \frac{z^2+5z+3}{(z-2)^{1+1}} dz$

CIF $\Rightarrow \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} \times f^n(z_0)$.

$\therefore \oint_c \frac{z^2+5z+3}{(z-2)^{1+1}} = \frac{2\pi i}{1!} \times f'(z_0)$, where $f(z) = z^2+5z+3$

$f'(z) = 2z+5$

$f'(z) = \quad =$

$z_0 = 2$

$f'(z_0) \Rightarrow 9$

$= 2\pi i \times 9$

$= 18\pi i$

a) Evaluate $\oint_c \frac{e^z}{z(1-z)^3} dz$; c is $|z| = \frac{1}{2}$

Soln:

$z = 0, 1$ are singularities. ; c ; $|z| = \frac{1}{2}$

when $z=0$, $|z| = 0 < \frac{1}{2} \Rightarrow$ lies inside c.

when $z=1$, $|z| = 1 > \frac{1}{2} \Rightarrow$ lies outside c.

$\therefore$ $z=0$ lies inside c and $z=1$ lies outside c.

$\therefore$ By CIF, $\oint_c \frac{f(z)}{z-z_0} dz = 2\pi i \times f(z_0)$

$z_0 = 0$

$\therefore \oint_c \frac{e^z}{z(1-z)^3} dz = \oint_c \frac{\left(e^z/(1-z)^3\right)}{z-0} dz = 2\pi i \times f(0)$

where $f(z) = \frac{e^z}{(1-z)^3}$

$f(z_0) = f(0) = \frac{e^0}{1-0} = \underline{\underline{1}}.$

$\therefore \oint_c \frac{e^z}{z(1-z)^3} dz = 2\pi i \times 1 = \underline{\underline{2\pi i}}$

---

2) Evaluate $\oint_c \frac{z^2+2z+3}{z^2-1} dz$; c is $|z-1| = 1$

Soln:

Here $z = \pm 1$ are singular points.

$(z^2-1 = (z+1)(z-1))$

when $z=+1$, $|z-1| = |1-1| = 0 < 1$ lies inside

when $z=-1$, $|z-1| = |-1-1| = 2 > 1$ lies outside

$\therefore$ $z=+1$ lies inside c and $z=-1$ lies outside c.

$\therefore$ By CIF, $\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i \times f(z_0)$

$$\Rightarrow \oint_c \frac{z^2+2z+3}{(z^2-1)}\,dz = \oint_c \frac{z^2+2z+3}{(z+1)(z-1)} = \oint_c \frac{\left(z^2+2z+3/z+1\right)}{z-1}$$

$$\Rightarrow \oint_c \frac{\left(\frac{z^2+2z+3}{z+1}\right)}{z-1}\,dz = 2\pi i \times f(z_0)$$

$$= 2\pi i \times f(1)$$

$$= 2\pi i \times 3$$

$$= \underline{\underline{6\pi i}}$$

$f(z) = \left(\frac{z^2+2z+3}{z+1}\right)$

$\therefore f(1) = \frac{6}{2} = 3$

Q) Using CIF, evaluate $\oint_c \frac{z+1}{z^4+2iz^3}\,dz$; $C; |z|=1$

Soln:

$$z^4+2iz^3 = 0 \Rightarrow z^3(z+2i) = 0$$

$$\Rightarrow z=0 \text{ and } z=-2i$$

$z=0, -2i$ are singularities.

$|a+ib| = \sqrt{a^2+b^2}$

when $z=0$, $|z| = |0| = 0<1$

when $z=-2i$, $|z| = |-2i| = \sqrt{4} = 2>1$

$\therefore$ $z=0$ lies inside c and $z=-2i$ lies outside c.

$\therefore$ By CIF, $\oint_c \frac{z+1}{z^4+2iz^3}\,dz = \oint_c \frac{z+1}{z^3(z+2i)}\,dz$

$$= \oint_c \frac{\left(z+1/z+2i\right)}{(z-0)^3}\,dz = \oint_c \frac{\left(\frac{z+1}{z+2i}\right)}{(z-0)^{2+1}}\,dz$$

$$= \frac{2\pi i}{2!} \times f''(z_0)$$

where $f(z) = \frac{z+1}{z+2i}$

$$f'(z) = \frac{(z+2i)\cdot 1 - (z+1)\cdot 1}{(z+2i)^2} = \frac{2i-1}{(z+2i)^2}$$

$$f''(z) = \frac{d}{dz}\frac{(2i-1)}{(z+2i)^2}$$

$$= (2i-1) \times \frac{-1}{(z+2i)^4} \times 2(z+2i)$$

$$\Rightarrow f''(z) = (2i-1)\cdot\frac{-2}{(z+2i)^3}, \quad z_0 = 0$$

$$\Rightarrow f''(0) = (2i-1)\cdot\frac{-2}{(2i)^3} = (2i-1)\cdot\frac{-2}{8\times -i}$$

$$\Rightarrow f''(0) = \frac{(2i-1)}{4i}$$

$$\therefore \oint_c \frac{\left(z+1/z+2i\right)}{(z-0)^3}\,dz = \frac{2\pi i}{2!}\times f''(0)$$

$$= \frac{2\pi i}{2!}\times\frac{(2i-1)}{4i}$$

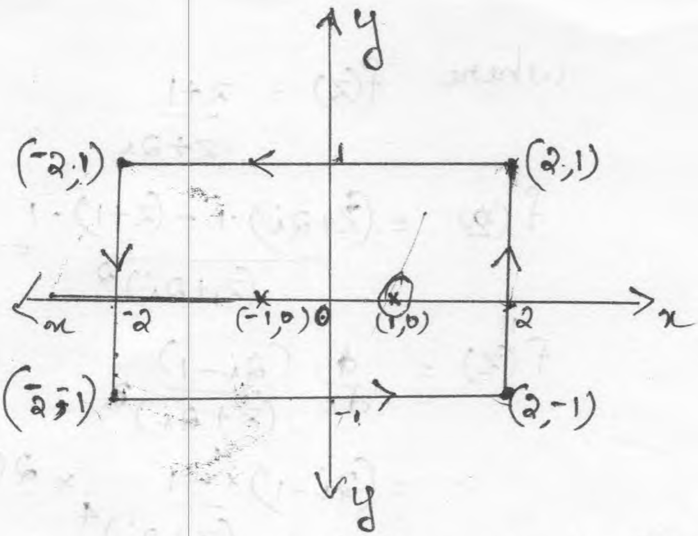
$$= \frac{\pi}{4}(2i-1)$$

Q) Evaluate $\oint_c \frac{\cos\pi z}{z^2-1}\,dz$; c is the rectangle with vertices $2\pm i$, $-2\pm i$.

Soln:

$$z^2-1=0 \Rightarrow (z+1)(z-1)=0 \Rightarrow z=-1, z=1$$

$\therefore z = \pm 1$ are singularities.



Here both the points $z = +1$ and $z = -1$ lies inside c.

$$\therefore \frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$$

Using partial fractions,

$$\frac{1}{(z+1)(z-1)} = \frac{A}{(z+1)} + \frac{B}{(z-1)}$$

$$\Rightarrow 1 = A(z-1) + B(z+1)$$

put $z=1 \Rightarrow 1 = 2B$

$$\Rightarrow \boxed{B = \frac{1}{2}}$$

put $z=-1 \Rightarrow 1 = -2A$

$$\Rightarrow \boxed{A = \frac{-1}{2}}$$

$$\therefore \frac{1}{z^2-1} = \frac{-1/2}{z+1} + \frac{1/2}{z-1}$$

$$\therefore \frac{\cos\pi z}{z^2-1} = -\frac{1}{2}\frac{\cos\pi z}{z+1} + \frac{1}{2}\cdot\frac{\cos\pi z}{z-1}$$

$$\oint_c \frac{\cos\pi z}{z^2-1}\,dz = \frac{-1}{2}\oint_c \frac{\cos\pi z}{z+1}\,dz + \frac{1}{2}\oint_c \frac{\cos\pi z}{z-1}\,dz$$

$$= \left[\frac{-1}{2} \times 2\pi i \times f(-1)\right] + \left[\frac{1}{2} \times 2\pi i \times f(1)\right]$$

where $f(z) = \cos \pi z$

$f(-1) = \cos \pi = (-1) = -1$

$f(1) = \cos \pi = -1$

$$\therefore \oint_c \frac{\cos \pi z}{z^2-1}\, dz = \frac{-1}{2} \times 2\pi i \times (-1) + \frac{1}{2} \times 2\pi i \times (-1)$$

$$= \pi i - \pi i$$

$$= \underline{\underline{0}}$$

Q) Using CIF, Calculate $\oint_c \frac{~~4-3z~~\ z^2}{z^3 - z^2 - z + 1}\, dz$

where C is (i) $|z+1| = \frac{3}{2}$

ii) $|z-1-i| = \frac{\pi}{2}$

Soln:

$z^3 - z^2 - z + 1 = 0 \Rightarrow z^2(z-1) - (z-1) = 0$

$\Rightarrow (z-1)[z^2 - 1] = 0$

$\Rightarrow (z-1)(z+1)(z-1) = 0$

$\therefore z = \pm 1$ are singular points.

i) $|z+1| = \frac{3}{2}$

when $z = 1$, $|z+1| = |2| = 2 > \frac{3}{2}$ lies outside c

when $z = -1$, $|z+1| = |-1+1| = 0 < \frac{3}{2}$ lies inside c.

$\therefore z = 1$ lies outside c and $z = -1$ lies inside c.

$$\therefore \oint_c \frac{z^2}{z^3 - z^2 - z + 1}\, dz = \oint_c \frac{z^2}{(z-1)^2 (z+1)}\, dz$$

$$\Rightarrow \oint_C \frac{\left(\frac{z^2}{(z-1)^2}\right)}{z+1} dz = 2\pi i \times f(z_0)$$

where $f(z) = \frac{z^2}{(z-1)^2}$, $z_0 = -1$

$$f(z_0) = f(-1) = \frac{1}{4}$$

$$\therefore \oint_C \frac{z^2/(z-1)^2}{z+1} dz = 2\pi i \times \frac{1}{4}$$

$$= \frac{\pi i}{2}$$

ii) $C;\ |z-1-i| = \frac{\pi}{2}$

when $z=1$, $|z-1-i| = |1-1-i| = |-i| = \sqrt{1} = 1 < \frac{\pi}{2}$

when $z=-1$, $|z-1-i| = |-1-1-i| = |-2-i| = \sqrt{-2^2+-1^2} = \sqrt{5} > \frac{\pi}{2}$

$\therefore z=1$ lies inside c and $z=-1$ lies outside c.

$$\therefore \oint_C \frac{z^2}{z^3-z^2-z+1} dz = \oint_C \frac{z^2}{(z-1)^2(z+1)} = \oint_C \frac{\left(z^2/z+1\right)}{(z-1)^2} dz.$$

By CIF $\Rightarrow$ $\oint \frac{\left(z^2/z+1\right)}{(z-1)^2} dz = \frac{2\pi i}{1!} \times f'(z_0)$, $z_0 = 1$

$$= 2\pi i \times f'(1)$$

$$= 2\pi i \times \frac{3}{4}$$

$$= \frac{3\pi i}{2}$$

where $f(z) = \frac{z^2}{z+1}$

$$f'(z) = \frac{(z+1)2z - z^2(1)}{(z+1)^2}$$

$$f'(z_0) = f'(1) = \frac{3}{4}$$

TAYLOR AND MACLAURIN SERIES

TAYLOR'S SERIES

If $f(z)$ is analytic in a closed curve c; $|z-a| = r$, then Taylor's series expansion of $f(z)$ at $z=a$ is

$$f(z) = f(a) + f'(a)\frac{(z-a)}{1!} + f''(a)\frac{(z-a)^2}{2!} + f'''(a)\frac{(z-a)^3}{3!} + \cdots$$

Maclaurin Series

at $a=0$, then Maclaurin series expansion of $f(z)$ is,

$$f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + f'''(0)\frac{z^3}{3!} + \cdots$$

※ Important Results

① $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

② $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$

③ $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

④ $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$

⑤ $(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \cdots$ if $|z|<1$

⑥ $(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \cdots$ if $|z|<1$

⑦ $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$ if $|z|<1$

⑧ $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$ if $|z|<1$

Q) Find the Taylor series expansion of $f(z) = \frac{1}{z^2}$ about $z=2$.

Soln:

Taylor's series expansion of $f(z)$ is,

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + \cdots$$

Here $f(z) = \frac{1}{z^2}$

at $z=2$, $f(z) = \frac{1}{z^2} = \frac{1}{(2+z-2)^2} = \frac{1}{2^2\left(1+\frac{z-2}{2}\right)^2}$

$$\Rightarrow f(z) = \frac{1}{4\left(1+\frac{z-2}{2}\right)^2}$$

$$= \frac{1}{4}\left[1+\frac{z-2}{2}\right]^{-2} \quad \text{using ⑦} \quad \left|\frac{z-2}{2}\right|<1$$

$$= \frac{1}{4}\left[1 - 2\left(\frac{z-2}{2}\right) + 3\left(\frac{z-2}{2}\right)^2 - 4\left(\frac{z-2}{2}\right)^3 + \cdots\right], \text{ if } \left|\frac{z-2}{2}\right|<1$$

$$= \frac{1}{4}\left[1 - (z-2) + 3\frac{(z-2)^2}{4} - 4\frac{(z-2)^3}{8} + \cdots\right] \text{ if } \left|\frac{z-2}{2}\right|<1$$

Since $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$ if $|z|<1$

Q) Find the Taylor series of $\frac{\sin z}{z-\pi}$ about the point $z=\pi$.

Soln:

put $z-\pi=t \Rightarrow z=\pi+t$

$$\therefore f(z) = \frac{\sin z}{z-\pi} = \frac{\sin(\pi+t)}{z-\pi} = \frac{-\sin t}{t} \quad \because \sin(\pi+\theta) = -\sin\theta$$

$$\therefore \Rightarrow \frac{-\sin t}{t} = \frac{-1}{t}\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots\right]$$

> Using the result,
> $$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\cdots$$

$$\Rightarrow f(z) = -1+\frac{t^2}{3!}-\frac{t^4}{5!}+\frac{t^6}{7!}-\cdots$$

$$\Rightarrow f(z) = -1+\frac{(z-\pi)^2}{3!}-\frac{(z-\pi)^4}{5!}+\frac{(z-\pi)^6}{7!}-\cdots, \quad |z-\pi|<1$$

Q) Find the Taylor series of $f(z)=\frac{z+1}{z-1}$ about $z=-1$.

Soln:

$$f(z) = \frac{z+1}{z-1}$$

$$= \frac{z+1}{z-1+1-1} = \frac{z+1}{z+1-2} = \frac{z+1}{-2\left(1-\frac{z+1}{2}\right)}$$

$$= \frac{z+1}{-2}\left(1-\frac{(z+1)}{2}\right)^{-1}$$

$$= \frac{z+1}{-2}\left[1+\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2+\left(\frac{z+1}{2}\right)^3+\cdots\right]$$

if $\left|\frac{z+1}{2}\right|<1$

$$= -\left[\frac{z+1}{2} + \frac{(z+1)^2}{2} + \frac{(z+1)^3}{3} + \frac{(z+1)^4}{4} + \cdots\right]$$

---

(2) Write the Maclaurin series expansion of the $f(z) = \frac{1}{z+1}$ at $z=0$

Soln:

Maclaurin series expansion of $f(z)$ is

$$f(z) = f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + f'''(0)\frac{z^3}{3!} + \cdots \quad — (1)$$

we've, $f(z) = \frac{1}{z+1}$, $f(0) = \frac{1}{0+1} = 1$

$$f'(z) = \frac{-1}{(z+1)^2}, \quad f'(0) = \frac{-1}{1} = -1$$

$$f''(z) = \frac{-1 \times -2}{(z+1)^3}, \quad f''(0) = \frac{2}{1^3} = 2$$

$$f'''(z) = \frac{-6}{(z+1)^4}, \quad f''(0) = \frac{-6}{1^4} = -6$$

Substituting all these values in $f(z)$ (1).

$$\therefore (1) \Rightarrow f(z) = 1 + -1.z + 2.\frac{z^2}{2!} + -6 \times \frac{z^3}{3!} + \cdots$$

$$\therefore \frac{1}{z+1} = f(z) = 1 - z + z^2 - z^3 + \cdots$$

MODULE V

RESIDUE INTEGRATION

LAURENT'S SERIES

If $f(z)$ is analytic within or on annulus, then for all z in that region,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

Note:

1. $(1+z)^{-1} = 1 - z + z^2 - \dots = \sum_{n=0}^{\infty} (-1)^n z^n$
2. $(1-z)^{-1} = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n$
3. $(1+z)^{-2} = 1 - 2z + 3z^2 - \dots = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n$
4. $(1-z)^{-2} = 1 + 2z + 3z^2 + \dots = \sum_{n=0}^{\infty} (n+1) z^n$

These expansions are valid if $|z|<1$.

Q) Expand $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in Laurents series for the region i) $|z|<2$ ii) $2<|z|<3$.

Soln:

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

Now $f(z) = \dfrac{z^2-1}{(z+2)(z+3)} = 1 + \dfrac{-5z-7}{(z+2)(z+3)}$

Now $\dfrac{-5z-7}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3}$

$\Rightarrow -5z-7 = A(z+3) + B(z+2)$

put $z=-3 \Rightarrow -B = 8 \Rightarrow \boxed{B=-8}$

put $z=-2 \Rightarrow A = 3 \Rightarrow \boxed{A=3}$

$\therefore \dfrac{-5z-7}{(z+2)(z+3)} = \dfrac{3}{z+2} + \dfrac{-8}{z+3}$

$\therefore f(z) = 1 + \dfrac{-5z-7}{(z+2)(z+3)}$

ie $f(z) = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

i) $|z|<2 \Rightarrow \left|\frac{z}{2}\right|<1$

$f(z) = 1 + \dfrac{3}{2\left(1+\frac{z}{2}\right)} - \dfrac{8}{3\left(1+\frac{z}{3}\right)}$

$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$\Rightarrow f(z) = 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n;$

This expansion is valid in $\left|\frac{z}{2}\right|<1$ and $\left|\frac{z}{3}\right|<1$.

$\Rightarrow |z|<2$ and $|z|<3$

$\Rightarrow |z|<2$

ii) $2 < |z| < 3 \Rightarrow |z| > 2$ and $|z| < 3$

$\Rightarrow \frac{2}{|z|} < 1$ and $\frac{|z|}{3} < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$|z| > 2$

$\Rightarrow \frac{|z|}{2} > 1$

$\Rightarrow \frac{2}{|z|} < 1.$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n;$$

This expansion valid only if, $\left|\frac{2}{z}\right| < 1$ and $\frac{|z|}{3} < 1$.

ie $2 < |z|$ and $|z| < 3$

ie $2 < |z| < 3$

Q) Find the Laurent's expansion of $\frac{1}{z-z^3}$ in $1 < |z+1| < 2$

Soln:

$$f(z) = \frac{1}{z-z^3} = \frac{1}{z(1-z^2)} = \frac{1}{z(1+z)(1-z)}$$

$$\frac{1}{z(1+z)(1-z)} = \frac{A}{z} + \frac{B}{1+z} + \frac{C}{1-z}$$

$$\Rightarrow 1 = A(1+z)(1-z) + Bz(1-z) + C(z)(1+z)$$

Put $z=0 \Rightarrow$ $\boxed{A=1}$

put $z=1 \Rightarrow 2C = 1 \Rightarrow$ $\boxed{C = \frac{1}{2}}$

put $z=-1 \Rightarrow 1 = -2B \Rightarrow \boxed{B = \frac{-1}{2}}$

$$\therefore f(z) = \frac{1}{z} + \frac{-1/2}{1+z} + \frac{1/2}{1-z} \quad ; \quad 1<|z+1|<2$$

$$\therefore f(z) = \frac{1}{(z+1-1)} + \frac{-1/2}{(1+z)} + \frac{1/2}{-(z+1-2)}$$

$$= \frac{1}{(u-1)} + \frac{-1/2}{(u)} + \frac{-1/2}{(u-2)} \quad ; \quad u = z+1$$

$$= \frac{1}{u-1} + \frac{-1}{2}\cdot\frac{1}{u} - \frac{1}{2}\cdot\frac{1}{u-2}$$

$$= \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{-1}{2}\cdot\frac{1}{u} - \frac{1}{2}\cdot\frac{1}{-2\left(1-\frac{u}{2}\right)}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{2u} + \frac{1}{4}\left(1-\frac{u}{2}\right)^{-1}$$

$$\therefore f(z) = \frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n - \frac{1}{2u} + \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{u}{2}\right)^n ;$$

valid in $\left|\frac{1}{u}\right|<1$ and $\left|\frac{u}{2}\right|<1$.

$\Rightarrow 1<|u|$ and $|u|<2$

$\Rightarrow 1<|u|<2$

$\Rightarrow 1<|z+1|<2$

$$\therefore f(z) = \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{1}{2(z+1)} + \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n ,$$

if $1<|z+1|<2$

Q) Find the Laurent's expansion of $\frac{1}{z(1-z)}$ valid in the region $|z+1| > 2$.

Soln:

$$f(z) = \frac{1}{z(1-z)} = \frac{A}{z} + \frac{B}{1-z}$$

$$\Rightarrow 1 = A(1-z) + Bz$$

When $z=0$, $1 = A \Rightarrow \boxed{A=1}$

When $z=1$, $1 = B \Rightarrow \boxed{B=1}$

$$\therefore f(z) = \frac{A}{z} + \frac{B}{1-z}$$

$$\text{ie } f(z) = \frac{1}{z} + \frac{1}{1-z}$$

$$= \frac{1}{z+1-1} + \frac{1}{-(z-1)}$$

$$= \frac{1}{z+1-1} + - \frac{1}{z+1-2}$$

$$= \frac{1}{u-1} - \frac{1}{u+2} \quad ; \text{Put } z+1 = u$$

$$= \frac{1}{u(1-\frac{1}{u})} - \frac{1}{u(1+\frac{2}{u})}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{1}{u}\left(1+\frac{2}{u}\right)^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^{n} - \frac{1}{u}\sum_{n=1}^{\infty}\left(\frac{2}{u}\right)^{n}$$

valid in $\left|\frac{1}{u}\right| < 1$ and $\left|\frac{2}{u}\right| < 1$

Q) Find the Laurent's series of $z^{-5} \sin z$ with centre '0'.

<u>soln</u>:

given, $f(z) = z^{-5} \sin z$

$\therefore \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$$= z^{-5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^4} - \frac{1}{z^2}\cdot\frac{1}{3!} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots$$

with centre '0'.

Q) Expand $f(z) = \frac{z}{(z+1)(z+2)}$ in Laurent's series about $z = -2$.

Soln:

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$z = A(z+2) + B(z+1)$$

put $z = -1 \Rightarrow -1 = A$

$$\Rightarrow \boxed{A = -1}$$

put $z = -2 \Rightarrow -2 = -B$

$$\Rightarrow \boxed{B = 2}$$

$$\therefore f(z) = \frac{A}{z+1} + \frac{B}{z+2}$$

$$= \frac{-1}{z+1} + \frac{2}{z+2}$$

given Region is about $z = -2$

ie $z+2$

$$\therefore f(z) = \frac{-1}{z+1} + \frac{2}{z+2} \text{ becomes}$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= \frac{-1}{-1[1-(z+2)]} + \frac{2}{z+2}.$$

$$= \frac{1}{[1-(z+2)]} + \frac{2}{(z+2)}.$$

$$= [1-(z+2)]^{-1} + \frac{2}{(z+2)}.$$

$$= \left[1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots\right] + \frac{2}{z+2}$$

$$= \left[\sum_{n=0}^{\infty} (z+2)^n\right] + \frac{2}{z+2}.$$

$$f(z) = \frac{2}{z+2} + \sum_{n=0}^{\infty} (z+2)^n \quad \text{about} \quad z = -2.$$

Q) Find the Laurent's series of $z^2 e^{1/z}$ with centre '0'.

Soln:

We've $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$\therefore$ for $e^{1/z} = 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!}$

But our given function is, $f(z) = z^2 e^{1/z}$

$$\therefore f(z) = z^2 e^{1/z}$$

$$= z^2 \left[1 + \frac{(\frac{1}{z})}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots\right]$$

$$= z^2 + \frac{z^1}{1!} + \frac{1}{2!} + \frac{1}{z} \cdot \frac{1}{3!} + \cdots$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6} z^{-1} + \cdots$$

$$\Rightarrow f(z) = (z-0)^2 + (z-0) + \frac{1}{2} + \frac{1}{6}(z-0)^{-1} + \cdots$$

with centre '0'.

Q) Find the Laurents series expansion of

$$f(z) = \frac{z}{(z^2-1)(z^2+4)}, \quad 1<|z|<2.$$

Soln:

$$f(z) = \frac{z}{(z^2-1)(z^2+4)} = \frac{z}{(z+1)(z-1)(z+2i)(z-2i)}$$

$$\Rightarrow \frac{z}{(z+1)(z-1)(z+2i)(z-2i)} = \frac{A}{z+1} + \frac{B}{(z-1)} + \frac{C}{(z+2i)} + \frac{D}{(z-2i)}$$

$$\Rightarrow z = A(z-1)(z+2i)(z-2i) + B(z+1)(z+2i)(z-2i) + C(z+1)(z-1)(z-2i) + D(z+1)(z-1)(z+2i)$$

Put $z=1$ $\Rightarrow 1 = B(1+1)(1+2i)(1-2i)$

$$\Rightarrow 1 = 2B(1+4)$$

$$\Rightarrow \boxed{B = 1/10}$$

Put $z=-1$ $\Rightarrow -1 = A(-1-1)(-1+2i)(-1-2i)$

$$\Rightarrow -1 = -2A(1+4)$$

$$\Rightarrow \boxed{A = 1/10.}$$

Put $z=2i$ $\Rightarrow D(2i+1)(2i-1)(2i+2i) = 2i$

$$\Rightarrow D(-4-1)\,4i = 2i$$

$$\Rightarrow \boxed{D = -1/10}$$

Put $z=-2i$ $\Rightarrow -2i = C(-2i+1)(-2i-1)(-2i-2i)$

$$\Rightarrow -2i = C(-4-1)\,-4i$$

$$\Rightarrow \boxed{C = -1/10}$$

$$\therefore f(z) = \frac{1/10}{z+1} + \frac{1/10}{z-1} + \frac{-1/10}{z+2i} + \frac{-1/10}{z-2i}$$

given region is $1<|z|<2$

$\Rightarrow 1<|z|$ and $|z|<2$

$\Rightarrow \frac{1}{|z|}<1$ and $\left|\frac{z}{2}\right|<1$

$$\therefore f(z) = \frac{1}{10}\cdot\frac{1}{z(1+\frac{1}{z})} + \frac{1}{10}\cdot\frac{1}{z(1-\frac{1}{z})} + \frac{-1}{10}\cdot\frac{1}{2i(1+\frac{z}{2i})} + \frac{-1}{10}\cdot\frac{1}{2i(1-\frac{z}{2i})}$$

$$= \frac{1}{10}\cdot\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} + \frac{1}{10}\cdot\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{20i}\left(1+\frac{z}{2i}\right)^{-1} + \frac{1}{20i}\left(1-\frac{z}{2i}\right)^{-1}$$

$$= \frac{1}{10}\cdot\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n + \frac{1}{10}\cdot\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{20i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2i}\right)^n + \frac{1}{20i}\sum_{n=0}^{\infty}\left(\frac{z}{2i}\right)^n$$

valid in $\frac{1}{|z|}<1$ and $\left|\frac{z}{2i}\right|<1$

ie $\frac{1}{|z|}<1$ and $\left|\frac{z}{2}\right|<1$

ie $1<|z|<2$.

CLASSIFICATION OF SINGULARITIES

If $f(z)$ is not analytic at $z=a$, then $z=a$ is called singular point or singularity of $f(z)$.

Eg: $f(z) = \frac{1}{z+2}$; $z=-2$ is singular point.

$f(z) = \frac{1}{z(z+1)(z+2)}$; $z = 0, -1, 2$ are singular points.

(1) ISOLATED SINGULAR POINT :-

A singular point $z=a$ of a function $f(z)$ is called an Isolated singular point if there exist a circle with centre at 'a' which contains no other singular points of $f(z)$.

Eg: 1) $f(z) = \frac{z}{z^2-1}$; $z = \pm 1$ are ~~sing~~ Isolated singular points.

2) $f(z) = \frac{1}{\sin \pi z}$; $\sin \pi z = 0 \Rightarrow \pi z = \pm n\pi$

$\Rightarrow z = \pm n$ $\quad \because \sin n\pi = 0$

$\therefore f(z)$ has infinite no. of isolated singular points.

3) $f(z) = \tan z$; $\Rightarrow f(z) = \frac{\sin z}{\cos z}$; $\cos z = 0$

$\Rightarrow z = \pm(2n+1)\frac{\pi}{2}$

$\because \cos(2n+1)\frac{\pi}{2} = 0.$

$\Rightarrow \quad z = \pm(2n+1)\frac{\pi}{2}$

$\therefore f(z)$ has infinitely many isolated singularity.

4) NON- ISOLATED SINGULAR POINT

Eg: $f(z) = \tan\left(\frac{1}{z}\right) = \frac{\sin(1/z)}{\cos(1/z)}$; $\cos\left(\frac{1}{z}\right) = 0$

$$\Rightarrow \frac{1}{z} = \pm(2n+1)\frac{\pi}{2}$$

$$\Rightarrow z = \frac{\pm 2}{(2n+1)\pi}$$

Here $z = 0$ is a non-isolated singular point.

(2) REMOVABLE SINGULARITIES

Suppose $z=a$ is an isolated singular point of $f(z)$ then we can express $f(z)$ as a Laurent's series about $z=a$ as,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots + \cdots$$
$$+ a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots\cdots$$

The first series is called Taylor part and the second series is called principal part. If the principal part is absent, $z=a$ is called removable singular point.

~~Eg~~ (3) ESSENTIAL SINGULARITY

If there are infinite no. of terms in the principal part, $z = a$ is called an Essential singularity.

(4) POLE

If there are finite no. of terms in the principal part, $z = a$ is called pole.

The highest power of $\frac{1}{z-a}$ is called ~~pole~~ order.

pole of order 1 is called simple pole.

Q) Determine and classify the singular points of the following functions.

1) $f(z) = \frac{1}{(z-3)(z-1)}$

$z = 3$ and $z = 1$ are isolated singularities.

Both are simple poles.

---

2) $f(z) = \cot\left(\frac{\pi}{z}\right)$

$$= \frac{\cos(\pi/z)}{\sin(\pi/z)}$$

$\sin \frac{\pi}{z} = 0$

$\Rightarrow \frac{\pi}{z} = \pm n\pi$

$$\Rightarrow z = \pm \frac{1}{n}$$

Here $z=0$ is a non-isolated singularity.

and $z = \pm \frac{1}{n}$ are isolated singularities.

---

3) $$f(z) = e^{1/z} = 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2}\cdot\frac{1}{2!} + \frac{1}{z^3}\cdot\frac{1}{3!} + \cdots$$

Here $z=0$ is a singular point. Since the principal part contains infinite no. of terms.

$\therefore$ $z=0$ is Essential singularity.

---

4) $$f(z) = \frac{1}{(z-3)^3(z+6)}$$

Here $z=3$ and $z=-6$ are singularities.

$z=3$ is a pole of order 3. and

$z=-6$ is a pole of order 1. ie simple pole.

---

5) $$f(z) = \frac{z - \sin z}{z^3}$$

$z=0$ is a singular point.

ie $$f(z) = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right]$$

$\therefore f(z) = \frac{1}{z^3}\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right]$

$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} + \cdots$

Since the principal part is absent, $z=0$ is a removable singularity.

---

6) $f(z) = \tan z = \frac{\sin z}{\cos z}$

$\Rightarrow \cos z = 0 \Rightarrow z = \pm(2n+1)\frac{\pi}{2}$

all are isolated singular points.

---

7) $f(z) = e^{-1/z^2}$

$f(z) = e^{-1/z^2} = 1 - \frac{(1/z^2)}{1!} + \frac{(1/z^2)^2}{2!} \mp \cdots$

$= 1 - \frac{1}{z^2} + \frac{1}{2!\,z^4} + \cdots$

Here $z=0$ is singularity.

Since principal part contains infinite no. of terms, $z=0$ is Essential singularity.

---

* 8) $f(z) = \frac{\sin z}{(z-\pi)^2}$

$\Rightarrow f(z) = \dfrac{\sin z}{(z-\pi)^2}$ $\quad \because \sin(\pi+\theta) = -\sin\theta$

$\Rightarrow z = \pi$ is a singular point.

$\therefore f(z) = \dfrac{\sin z}{(z-\pi)^2} = \dfrac{\sin(z-\pi+\pi)}{(z-\pi)^2}$

$= \dfrac{\sin[\pi + (z-\pi)]}{(z-\pi)^2} = \dfrac{-\sin(z-\pi)}{(z-\pi)^2}$

$= -\dfrac{1}{(z-\pi)^2}\left[(z-\pi) - \dfrac{(z-\pi)^3}{3!} + \dfrac{(z-\pi)^5}{5!} - \cdots\cdots\right]$

$\quad \because \sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots$

$f(z) = -\dfrac{1}{(z-\pi)} + \dfrac{(z-\pi)}{3!} - \dfrac{(z-\pi)^3}{5!} + \cdots$

Here principal part contains only one term ie $\left(\frac{1}{z-\pi}\right)$ term. ie finite no. of term in principal part.

$\therefore$ $z = \pi$ is a pole of order '1'.

ie $z = \pi$ is a simple pole.

(9) $f(z) = e^{\frac{1}{z-1}}$

$$\therefore f(z) = e^{\frac{1}{z-1}} = 1 + \frac{\left(\frac{1}{z-1}\right)}{1!} + \frac{\left(1/z-1\right)^2}{2!} + \frac{\left(1/z-1\right)^3}{3!} + \cdots$$

$$= 1 + \frac{1}{(z-1)} + \frac{1}{2!}\frac{1}{(z-1)^2} + \frac{1}{3!}\frac{1}{(z-1)^3} + \cdots$$

$\Rightarrow z = 1$ is a singular point.

Since principal part contains infinite no. of terms, $z=1$ is an Essential singularity.

ie $z=1$ is Essential singular point.

---

(10) $f(z) = \dfrac{1 - \cos z}{z^2}$

$z = 0$ is a singular point.

$$f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right]$$

$$\left\langle \therefore \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \right.$$

$$= \frac{1}{z^2}\left[\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right]$$

$$= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots$$

Here principal part is absent.

$\therefore$ $z=0$ is Removable singularity.

(ii) $f(z) = \dfrac{\sin z}{z^4}$

Here $z=0$ is a singular point.

$$\therefore f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \right]$$

$$= \frac{1}{z^3} - \frac{1}{3!}\cdot\frac{1}{z} + \frac{1}{5!}z - \frac{1}{7!}z^3 + \cdots$$

Here principal part contains finite no. of terms, ie only one term $\frac{1}{z^3}$.

$\therefore$ $z=0$ is a pole of order 3.

RESIDUE OF A FUNCTION

Residue of a function $f(z)$ at $z=a$ is the Coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion of $f(z)$ and is denoted by $\underset{z=a}{\text{Res}}\, f(z)$.

CALCULATION OF RESIDUE.

① If $f(z)$ has a simple pole at $z=a$,

then $$\underset{z=a}{\text{Res}}\, f(z) = \lim_{z\to a} (z-a)\, f(z)$$

or

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)}$$, where $f(z) = \frac{g(z)}{h(z)}$ and $g(a) \neq 0$.

② If $f(z)$ has a pole of order m at $z=a$,

then $$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$, $m \geq 2$

Q) Find the singularities and Residue of $f(z) = \frac{1}{z^4-1}$

Soln:

$$f(z) = \frac{1}{z^4-1} = \frac{1}{(z^2+1)(z^2-1)} = \frac{1}{(z+1)(z-1)(z+i)(z-i)}$$

The singularities are $z = 1, -1, i, -i$.

all are simple poles.

at $z=1$,

$$\underset{z=1}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)} = \frac{g(1)}{h'(1)}$$ where $g(z) = 1$

$g(a) = g(1) = 1$

$$= \frac{1}{4}$$

also $h(z) = z^4-1$

$h'(z) = 4z^3$

$h'(a) = h'(1) = 4$

at $z=-1$,

$$\underset{z=-1}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)} = \frac{g(-1)}{h'(-1)}$$

$g(a) = g(1) = 1$

$$= \frac{1}{-4}$$

$h'(a) = h'(-1) = -4$

at $z=i$,

$$\underset{z=i}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)} = \frac{g(i)}{h'(i)}$$

$g(a) = g(i) = 1$

$$= \frac{1}{-4i}$$

$h'(a) = h'(i)$

$= 4i^3$

$$= \frac{i}{4}$$ $\angle$ since $\frac{1}{i} = -i$

$= -4i$

at $z = -i$

$$\underset{z=-i}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)} = \frac{g(-i)}{h'(-i)}$$

$$g(a) = g(-i) = 1$$

$$h'(a) = h'(-i) = 4(-i)^3 = 4 \times -1 \times i^3 = 4i$$

$$= \frac{1}{4i}$$

$$= \frac{-i}{4} \quad \left(\frac{1}{i} = -i\right)$$

Q) Find the residue of $f(z) = \dfrac{ze^z}{z^2+4}$

Soln:

$z^2+4 = 0 \Rightarrow (z+2i)(z-2i) = 0$

$\therefore$ Singular points are $z = +2i$ and $z = -2i$

both are simple poles.

$$\therefore f(z) = \frac{ze^z}{z^2+4} = \frac{ze^z}{(z+2i)(z-2i)}$$

To find residue,

Residue at $z = 2i$,

$$\underset{z=2i}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)} \quad \text{where } g(z) = ze^z$$

$$g(a) = g(2i) = 2ie^{2i}$$

$$h(z) = z^2+4$$

$$h'(z) = 2z$$

$$h'(a) = h'(2i) = 4i$$

$$= \frac{2ie^{2i}}{4i}$$

$$= \frac{e^{2i}}{2}$$

Residue at $z = -2i$,

$$\underset{z=-2i}{\text{Res}}\, f(z) = \frac{g(a)}{h'(a)}$$

$$= \frac{-2i\,e^{-2i}}{-4i}$$

$$= \frac{e^{-2i}}{2}.$$

where $g(z) = z e^{z}$

$g(a) = g(-2i) = -2i\,e^{-2i}$

$h(z) = z^2 + 4$

$h'(z) = 2z$

$h'(a) = h'(-2i) = -4i$

Q) Find the residue and singular points of $f(z) = \tan z$.

Soln:

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

To find singular points, $\cos z = 0$

$$\Rightarrow z = \pm(2n+1)\frac{\pi}{2}$$

when $n = 0, 1, \ldots$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$

$\Rightarrow$ simple poles.

All are simple poles.

Residue at $z = \pm(2n+1)\frac{\pi}{2}$ is,

$$\underset{z=\pm(2n+1)\frac{\pi}{2}}{\text{Res}}\, f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=\pm(2n+1)\frac{\pi}{2}}$$

where $f(z) = \frac{g(z)}{h(z)}$

$g(z) = \sin z$

$h(z) = \cos z$

$$= \left.\frac{\sin z}{-\sin z}\right|_{z \pm (2n+1)\frac{\pi}{2}}$$

$$= -1.$$

2) Find the residue of $f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$

soln:

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} = \frac{z^2 - 2z}{(z+1)^2 (z+2i)^1 (z-2i)^1}$$

Here singularities are,

$z = -1$ is a pole of order 2.

$z = -2i$ is a pole of order 1. ie simple pole.

$z = 2i$ is a pole of order 1. ie simple pole.

$$\underset{z=-1}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z) \right], \quad (m=2)$$

$$= \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d^{2-1}}{dz^{2-1}} \left[(z+1)^2 \cdot \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} \right]$$

$$= \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} \left[\frac{z^2 - 2z}{z^2+4} \right]$$

$$= \lim_{z \to -1} \left[\frac{(z^2+4)(2z-2) - (z^2-2z)\, 2z}{(z^2+4)^2} \right]$$

$$= \left[\frac{((-1)^2+4)(2 \times -1 - 2) - ((-1)^2 - 2 \times -1) \cdot 2 \times -1}{[(-1)^2+4]^2} \right]$$

$$= \left[\frac{(5 \times -4) + 6}{25} \right] = \frac{-14}{25}$$

$$\therefore \underset{z=-1}{\text{Res}}\, f(z) = \frac{-14}{25} \; //.$$

$z = 2i$ is a simple pole.

Now $\underset{z=2i}{\text{Res}}\, f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=2i}$

$= \dfrac{g(a)}{h'(a)} = \dfrac{g(2i)}{h'(2i)}$ where,

$g(z) = z^2 - 2z$

$\therefore g(a) = g(2i)$

$= (2i)^2 - 2\times 2i$

$\therefore g(2i) = -4 - 4i$

and $h(z) = (z+1)^2 (z^2+4)$

$h'(z) = (z+1)^2 \times 2z + (z^2+4)\times 2(z+1)$

$h'(a) = h'(2i) =$

$= (2i+1)^2 \times 2\times 2i + ((2i)^2 + 4)\times 2(2i+1)$

$= (2i+1)^2 . 4i + 0$

$\therefore h'(2i) = 4i(2i+1)^2$

$\therefore \underset{z=2i}{\text{Res}}\, f(z) = \dfrac{(-4-4i)}{4i(2i+1)^2}$

$= \dfrac{-4-4i}{4i(-4+4i+1)}$

$= \dfrac{-4-4i}{4i(-3+4i)}$

$= \dfrac{-4-4i}{4i(4i-3)}$

Now Residue of $f(z)$ at $z = -2i$.

$z = -2i$ is a simple pole.

$\therefore \underset{z=-2i}{\text{Res}}\, f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=-2i}$

$= \left.\dfrac{z^2 - 2z}{(z+1)^2 \times 2z + (z^2+4)\times 2(z+1)}\right|_{z=-2i}$

$= \dfrac{(-2i)^2 - 2\times(-2i)}{(-2i+1)^2 \times 2\times(-2i) + ((-2i)^2+4)\times 2(-2i+1)}$

$$\therefore \operatorname*{Res}_{z=-2i} f(z) = \frac{-4+4i}{(-2i+1)^2 \times -4i + 0}$$

$$= \frac{-4+4i}{(-4+-4i+1) \times -4i}$$

$$\therefore \operatorname*{Res}_{z=-2i} f(z) = \frac{-4+4i}{(-4i-3) \cdot -4i}$$

RESH

Q) Calculate Residue of $f(z) = \dfrac{1}{z^2(1+z)}$

Soln:

singularities are $z=0$ and $z=-1$.

$z=0$ is a pole of order 2.

$z=-1$ is a pole of order 1.

Residue at $z=0$ (order 2, ie $m=2$)

$$\therefore \operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z) \right]$$

$$\text{ie } \operatorname*{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d^{2-1}}{dz^{2-1}} \left[\frac{(z-0)^2}{z^2(1+z)} \right]$$

$$= \lim_{z \to 0} \frac{d}{dz} \left[\frac{1}{1+z} \right].$$

$$= \lim_{z \to 0} \left[\frac{-1}{(1+z)^2} \right] = \frac{-1}{(1+0)^2} = -1$$

ie $\underset{z=0}{\text{Res}}\ f(z) = -1$

Now Residue at $z=-1$. (pole of order 1. simple pole.)

$\therefore \underset{z=a}{\text{Res}}\ f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=a}$

ie $\underset{z=-1}{\text{Res}}\ f(z) = \left.\dfrac{1}{z^2 \cdot 1+(1+z)2z}\right|_{z=-1}$

$= \dfrac{1}{(-1)^2+(1+-1)\cdot 2\cdot -1}$

$= \dfrac{1}{1}$

$= \underline{\underline{1}}$

$\therefore \underset{z=-1}{\text{Res}}\ f(z) = \underline{\underline{1}}$

where,

$g(z) = 1$

$\therefore g(a) = g(-1) = 1$

~~$g(z)\ h(z) = \dfrac{1}{z^2(1+z)}$~~

$h(z) = z^2(1+z)$

$h'(z) = z^2\cdot 1+(1+z)2z$

$\therefore h'(-1) = -1^2\cdot 1+(1+-1)\times 2$

$= \underline{\underline{1}}$.

Q) Find the Residue of $f(z) = \dfrac{z^3+2z}{(z-i)^3}$

Soln:-

Singularities are $z=i$ (order 3. ie $m=3$).

$\therefore \underset{z=a}{\text{Res}}\ f(z) = \dfrac{1}{(m-1)!} \lim_{z\to a} \dfrac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m \cdot f(z)\right]$.

ie $\underset{z=i}{\text{Res}}\, f(z) = \frac{1}{(3-1)!} \lim_{z\to i} \frac{d^{3-1}}{dz^{3-1}} \left[(z-i)^3 \cdot \frac{z^3+2z}{(z-i)^3} \right]$

$= \frac{1}{2!} \lim_{z\to i} \frac{d^2}{dz^2} \left[z^3+2z \right]$

$= \frac{1}{2} \lim_{z\to i} \left[\frac{d}{dz} \left(3z^2+2 \right) \right]$

$= \frac{1}{2} \lim_{z\to i} \left[6z \right]$

$= \frac{1}{2} \lim_{z\to i} \left[6z \right]$

$= \frac{1}{2} \left[6 \cdot i \right] = 3i$

$\therefore \underset{z=i}{\text{Res}}\, f(z) = \underset{z=i}{\text{Res}}\, \frac{z^3+2z}{(z-i)^3} = 3i$

CAUCHY'S RESIDUE THEOREM (CRT)

Evaluation of Integral

If f(z) is analytic at all points inside and on a simple closed curve C except at a finite number of isolated singular points within C, then

$$\oint_C f(z)\, dz = 2\pi i \left[\text{sum of residues at singular points within } C \right]$$

Q) Evaluate $\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz$; C is $|z|=3$.

Using Cauchy's Residue theorem.

Soln:

Here $z=1$, and $z=2$ are singularities.

both are simple poles.

both lies inside $C ; |z|=3$

$\left[\begin{array}{l} \text{ie } |1| = 1 < 3 \text{ and} \\ |2| = 2 < 3. \end{array}\right] \Rightarrow$ lies inside C.

∴ By Cauchy's Residue theorem,

$\oint_C f(z)\, dz = 2\pi i$ [Sum of residues at points inside c.]

$= 2\pi i \left[\underset{z=1}{\text{Res}}\, f(z) + \underset{z=2}{\text{Res}}\, f(z) \right]$

Now $\underset{z=1}{\text{Res}}\, f(z) = \left. \frac{g(z)}{h'(z)} \right|_{z=1}$ $\quad h(z) = (z-1)(z-2)$

$= \left. \frac{\cos \pi z^2}{(z-1)\cdot 1 + (z-2)\cdot 1} \right|_{z=1}$

$= \frac{\cos \pi}{0 + -1 \cdot 1} = \frac{\cos \pi}{-1} = \frac{-1}{-1}$

ie $\underset{z=1}{\text{Res}}\, f(z) = \underline{\underline{1}}$.

Now $\underset{z=2}{\text{Res}}\, f(z) = \left.\dfrac{g(z)}{h'(z)}\right|_{z=2}$

$$= \left.\frac{\cos\pi z^2}{(z-1)\cdot 1+(z-2)\cdot 1}\right|_{z=2}$$

$$= \frac{\cos\pi(4)}{1+0}$$

$$= \frac{\cos 4\pi}{1} = \frac{1}{1} \qquad \because \cos 4\pi = 1 \text{ ; } 4 \text{ is even.}$$

ie $\underset{z=2}{\text{Res}}\, f(z) = \underline{\underline{1}}$

$\therefore$ By CRT,

$$\oint_C f(z)\,dz = 2\pi i\left[\underset{z=1}{\text{Res}}\, f(z) + \underset{z=2}{\text{Res}}\, f(z)\right]$$

$$= 2\pi i\,[+1+1]$$

$$\therefore \oint_C \frac{\cos\pi z^2}{(z-1)(z-2)}\,dz = \underline{\underline{4\pi i}}.$$

Q) Use Residue Theorem to evaluate $\oint_C \frac{30z^2 - 23z + 5}{(2z-1)^2(3z-1)} dz$

where c ; $|z| = 1$

Soln:

$(2z - 1) = 0 \Rightarrow z = \frac{1}{2}$

$(3z - 1) = 0 \Rightarrow z = \frac{1}{3}$

$z = 1/2$ and $z = 1/3$ are singularities.

$|z| = |\frac{1}{2}| = \frac{1}{2} < 1$ lies inside c.

$|z| = |\frac{1}{3}| = \frac{1}{3} < 1$ lies inside c.

both points lies inside c.

$\therefore$ By Cauchy's Residue Theorem,

$$\oint_C f(z)\, dz = 2\pi i \left[\text{sum of Residues at singularities inside c.} \right]$$

$$= 2\pi i \left[\underset{z=\frac{1}{2}}{\text{Res}} f(z) + \underset{z=\frac{1}{3}}{\text{Res}} f(z) \right]$$

Now Residue at $z = \frac{1}{2}$, $\left[z = \frac{1}{2} \text{ is a pole of order } 2 \right]$.

, $m = 2$.

$$\underset{z=\frac{1}{2}}{\text{Res}} f(z) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

$$= \frac{1}{(2-1)!} \lim_{z \to \frac{1}{2}} \frac{d^1}{dz} \left[\left(z - \frac{1}{2}\right)^2 \cdot \frac{(30z^2 - 23z + 5)}{(2z-1)^2 (3z-1)} \right]$$

$$= \frac{1}{1!} \lim_{z \to \frac{1}{2}} \frac{d}{dz} \left[\left(z - \frac{1}{2}\right)^2 \cdot \frac{(30z^2 - 23z + 5)}{4\left(z - \frac{1}{2}\right)^2 (3z-1)} \right]$$

$$= \frac{1}{4} \lim_{z \to \frac{1}{2}} \left[\frac{(3z-1)(60z-23) - (30z^2-23z+5) \cdot 3}{(3z-1)^2} \right]$$

$$= \frac{1}{4} \left[\frac{\frac{1}{2} \cdot 7 - \left(\frac{15}{2} - \frac{23}{2} + 5 \right) \cdot 3}{1/4} \right]$$

$$= \frac{7}{2} - 3 = \frac{1}{2}$$

ie $\underset{z=\frac{1}{2}}{\text{Res}} f(z) = \frac{1}{2}$

Now Residue at $z = \frac{1}{3}$, ($\because z = \frac{1}{3}$ is a simple pole)

$$\underset{z=\frac{1}{3}}{\text{Res}} f(z) = \left. \frac{g(z)}{h'(z)} \right|_{z=\frac{1}{3}} = \frac{30z^2 - 23z + 5}{(2z-1)^2 \cdot 3 + (3z-1) \cdot 2(2z-1) \cdot 2}$$

$$= \frac{\frac{10}{3} - \frac{23}{3} + 5}{1/3 + 0} = \frac{2/3}{1/3} = 2$$

$\therefore \underset{z=\frac{1}{3}}{\text{Res}} f(z) = 2$

By CRT, $\oint_C f(z)\, dz = 2\pi i \left[\underset{z=1/2}{\text{Res}} f(z) + \underset{z=\frac{1}{3}}{\text{Res}} f(z) \right]$

$$= 2\pi i \left[1/2 + 2 \right]$$

$$= 5\pi i$$

2) Evaluate $\int \frac{\tan z}{z^2-1}\, dz$ $C;\ |z| = \frac{3}{2}$

Soln:

$z = \pm 1$ are singularities. simple poles.

$(z^2-1) = (z+1)(z-1) \Rightarrow z=1$ and $z=-1$ are simple poles

$C;\ |z| = \frac{3}{2}$

at $z=1 \Rightarrow |1| = 1 < \frac{3}{2}$

at $z=-1 \Rightarrow |-1| = 1 < \frac{3}{2}$ $\Rightarrow$ both lies inside C.

By CRT, $\oint_C f(z)\, dz = 2\pi i$ [Sum of Residues inside C.]

Now $\operatorname{Res}_{z=1} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=1}$

$= \left.\frac{\tan z}{2z}\right|_{z=1} = \frac{\tan 1}{2}$

and $\operatorname{Res}_{z=-1} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-1}$

$= \left.\frac{\tan z}{2z}\right|_{z=-1} = \frac{\tan(-1)}{-2}$

$= \frac{-\tan 1}{-2} = \frac{\tan 1}{2}$

$\therefore$ By CRT, $\oint_C f(z)\, dz = 2\pi i \left[\frac{\tan 1}{2} + \frac{\tan 1}{2} \right]$

$= 2\pi i \tan 1.$

Q) Evaluate $\oint_C \frac{z^2}{(z-1)^2(z-2)}$ where C ; $|z| = 2.5$ using Cauchy's Residue theorem.

Soln:

$$f(z) = \frac{z^2}{(z-1)^2(z-2)}$$

$|1| = 1 < 2.5$

$|2| = 2 < 2.5$

Singularities are:-

$z=1$ is a pole of order 2.

$z=2$ is a simple pole. $\Rightarrow$ both lies inside c.

By CRT, $\oint_C f(z)\,dz = 2\pi i \left[\underset{z=1}{\text{Res}}\, f(z) + \underset{z=2}{\text{Res}}\, f(z) \right]$

Now $\underset{z=1}{\text{Res}}\, f(z)$ (pole of order 2 ; $m=2$)

$$= \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

$$= \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d^1}{dz^1} \left[(z-1)^2 \cdot \frac{z^2}{(z-1)^2 \cdot (z-2)} \right]$$

$$= \lim_{z\to 1} \frac{d}{dz} \left[\frac{z^2}{z-2} \right]$$

$$= \lim_{z\to 1} \left[\frac{(z-2)\cdot 2z - z^2 \cdot 1}{(z-2)^2} \right]$$

$$= \frac{-1\cdot 2 - 1}{1} = \frac{-3}{1} = \underline{\underline{-3}}$$

$\therefore \underset{z=1}{\text{Res}}\, f(z) = -3$ //

Now $\underset{z=2}{\text{Res}}\, f(z)$ = (Simple pole)

$$= \left.\frac{g(z)}{h'(z)}\right|_{z=2} \qquad g(z) = z^2, \quad h(z) = (z-1)^2 \cdot (z-2)$$

$$= \left.\frac{z^2}{(z-1)^2 \cdot 1 + (z-2)\, 2(z-1) \cdot 1}\right|_{z=2}$$

$$= \frac{4}{1+0} = \underline{\underline{4}}$$

$$\therefore \oint f(z)\, dz = 2\pi i \left[\underset{z=1}{\text{Res}}\, f(z) + \underset{z=2}{\text{Res}}\, f(z) \right]$$

$$= 2\pi i \left[-3 + 4 \right]$$

$$= 2\pi i\,(1) = \underline{\underline{2\pi i}}$$

Q) Evaluate $\int \frac{z-3}{z^2+2z+5}\, dz$ where i) C is $|z| = 1$

using CRT.

ii) C ; $|z+1-i| = 2$

iii) C ; $|z+1+i| = 2$

Soln:

To find singularities, $z^2+2z+5 = 0$

$$\Rightarrow z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = \underline{\underline{-1 \pm 2i}}$$

The singularities are $z = -1+2i$ and

$z = -1-2i$

i) $C;\ |z|=1$

when $z=-1+2i$, $|z|=|-1+2i|=\sqrt{5}>1$, lies outside c

when $z=-1-2i$, $|z|=|-1-2i|=\sqrt{5}>1$, lies outside c.

∴ By Cauchy's Integral Theorem,

$$\oint_C f(z)\,dz = 0$$

ii) $C;\ |z+1-i|=2$

when $z=-1+2i$, $|z|=|(-1+2i)+1-i|=|i|=\sqrt{1}=1<2$, lies inside c.

when $z=-1-2i$, $|z|=|(-1-2i)+1-i|=|-3i|=\sqrt{9}=3>2$, lies outside c.

Also, $z=-1+2i$ is a simple pole.

$$\therefore \underset{z=-1+2i}{\text{Res}}\, f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-1+2i}$$

where $g(z)=z-3$, $h(z)=z^2+2z+5$

$$= \left.\frac{z-3}{2z+2}\right|_{z=-1+2i}$$

$$= \frac{(-1+2i)-3}{2(-1+2i)+2}$$

$$= \frac{-4+2i}{4i}$$

$$\therefore \underset{z=-1+2i}{\text{Res}}\, f(z) = \frac{-2+i}{2i}$$

$\therefore$ By Cauchy's Residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \left[\text{sum of residues at singular points within } C\right].$$

$$= 2\pi i \left[\underset{z=-1+2i}{\text{Res}} f(z) \right]$$

$$= 2\pi i \left[\frac{i-2}{2i} \right]$$

$$= \pi(i-2)$$

iii) C ; $|z+1+i| = 2$

when $z = -1+2i$, $|z+1+i| = |(-1+2i)+1+i| = |3i| = 3 > 2$, lies outside C.

when $z = -1-2i$, $|z+1+i| = |(-1-2i)+1+i| = |-i| = 1 < 2$, lies inside C.

Also $z = -1-2i$ is a simple pole.

$$\therefore \underset{z=-1-2i}{\text{Res}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-1-2i} \quad \text{where } g(z) = z-3, \; h(z) = z^2+2z+5$$

$$= \left.\frac{z-3}{2z+2}\right|_{z=-1-2i}$$

$$= \frac{(-1-2i)-3}{2(-1-2i)+2} = \frac{-4-2i}{-4i} = \frac{2+i}{2i}$$

$\therefore$ By CRT, $\oint_C f(z)\,dz = 2\pi i\left[\text{Sum of residues at singularities within } C\right]$

$$= 2\pi i \left[\frac{2+i}{2i}\right] = \pi(2+i)$$

Q) Find all singular points and corresponding residues of $f(z) = \frac{z+2}{(z+1)^2(z-2)}$

Soln:

put $(z+1)^2(z-2) = 0$ ∴ Singularities are, $z=-1$ is a pole of order 2.

$z=2$ is a simple pole.

Now Residue f(z) at $z=-1$ $(m=2)$

$$\therefore \operatorname{Res}_{z=-1} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m \cdot f(z)\right)$$

$$= \frac{1}{(2-1)!} \lim_{z\to -1} \frac{d}{dz} \left[\frac{(z+1)^2 \cdot (z+2)}{(z+1)^2 \cdot (z-2)} \right]$$

$$= \lim_{z\to -1} \frac{d}{dz} \left[\frac{z+2}{z-2} \right]$$

$$= \lim_{z\to -1} \left[\frac{(z-2)\cdot 1 - (z+2)\cdot 1}{(z-2)^2} \right]$$

$$= \left[\frac{(-1-2) - (-1+2)}{(-1-2)^2} \right] = \frac{-3-(1)}{(-3)^2} = \frac{-4}{9}$$

Now $\operatorname{Res}_{z=2} f(z)$ (simple pole)

$$= \left. \frac{g(z)}{h'(z)} \right|_{z=2}$$

$$= \left. \frac{z+2}{(z+1)^2 \cdot 1 + (z-2)\cdot 2(z+1)} \right|_{z=2}$$

$$= \frac{4}{3^2+0} = \frac{4}{9}$$

$$\therefore \underset{z=2}{\text{Res}}\, f(z) = \frac{4}{9}$$

Q) Find the residue of $\frac{e^z}{z^3}$ at its pole.

Soln:

$z=0$ is a pole of order 3. ie $m=3$.

$$\therefore \underset{z=0}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$= \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^{3-1}}{dz^{3-1}}\left[(z-0)^3 \cdot \frac{e^z}{z^3}\right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[e^z\right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \left[e^z\right]$$

$$= \frac{1}{2!} e^0$$

$$= \frac{1}{2}$$

$$\frac{d}{dz}(e^z) = e^z$$

$$\frac{d^2}{dz^2} e^z = e^z$$

Q) Using Cauchy's Residue theorem evaluate

$$\oint_C \frac{\cosh \pi z}{z^2+4} \quad ; \quad C : |z| = 3$$

Soln:

Singularities are $z^2+4 = 0$

$$\Rightarrow (z+2i)(z-2i) = 0$$

$$\Rightarrow z = \pm 2i ,$$

$\therefore$ Singular points are $z = +2i$ and $z = -2i$

both are simple poles.

when $z = 2i$; $|z| = |2i| = \sqrt{4} = 2 < 3$ lies inside c.

when $z = -2i$; $|z| = |-2i| = \sqrt{4} = 2 < 3$ lies inside c.

both points lies inside c.

Now $\underset{z=2i}{\text{Res}}\, f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=2i} =$

$$= \left.\frac{\cosh \pi z}{2z}\right|_{z=2i}$$

$$= \frac{\cosh \pi(2i)}{2 \cdot 2i} = \frac{\cosh 2\pi i}{4i} = \frac{1}{4i}$$

also $\underset{z=-2i}{\text{Res}}\, f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-2i}$

$$= \left.\frac{\cosh \pi z}{2z}\right|_{z=-2i}$$

$$= \frac{\cosh \pi(-2i)}{2 \cdot -2i} = \frac{\cosh -2\pi i}{-4i} = \frac{1}{-4i}$$

$\therefore$ By CRT, $\oint_C f(z)\,dz = 2\pi i \left[\underset{z=2i}{\text{Res}}\, f(z) + \underset{z=-2i}{\text{Res}}\, f(z) \right]$

$$= 2\pi i \left[\frac{1}{4i} + \frac{1}{-4i} \right]$$

$$= 0$$

Q) Evaluate $\oint_C \frac{z-23}{z^2-4z-5}\,dz$; $C: |z-i| = 2$ using Residue Integration theorem.

Soln:

$$z^2 - 4z - 5 = 0 \Rightarrow (z+1)(z-5) = 0.$$

$\Rightarrow z = -1$, and $z = 5$ are singularities, both are simple poles.

when $z = -1$; ~~$C: |z| = |-1| = 1 <$~~ $C: |z-i| = |-1-i| = \sqrt{2} < 2$; lies inside C.

when $z = 5$; $C: |z-i| = |5-i| = \sqrt{26} > 2$ lies outside C.

$\therefore$ By CRT, $\oint_C f(z)\,dz = 2\pi i$ [Sum of residues at singularity inside C].

$$= 2\pi i \left[\underset{z=-1}{\text{Res}}\, f(z) \right]$$

Now $\underset{z=-1}{\text{Res}}\, f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-1} = \left.\frac{z-23}{2z-4}\right|_{z=-1}$

$$= \frac{-24}{-6} = 4$$

$\therefore$ By CRT $\oint_C f(z)\,dz = 2\pi i \left[4 \right]$

$= 8\pi i$

APPLICATION OF RESIDUES TO EVALUATE REAL INTEGRALS

1. Integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function on $\sin\theta$ and $\cos\theta$.

This integral can be reduced to complex line integral by means of the substitution

$z = e^{i\theta}$ ie $|z| = 1$.

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{z^2-1}{2iz}$$

The integral can be evaluated by Cauchy's Residue theorem.

<u>Note</u>:

$$\int_0^{\pi} f(\cos\theta, \sin\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta.$$

Q) Using contour integration evaluate

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta$$

Soln:

$z = e^{i\theta}$; $|z| = 1$

$d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$

$$\therefore \int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta = \int_{C:|z|=1} \frac{1}{5+4\frac{(z^2+1)}{2z}} \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{1}{\frac{4z^2+10z+4}{2\cancel{z}}} \cdot \frac{dz}{\cancel{z}}$$

$$= \frac{1}{i}\int_C \frac{1}{2z^2+5z+2}\, dz. \qquad \text{——— (1)}$$

Singular points are, $2z^2+5z+2=0$

$$\Rightarrow z = \frac{-5 \pm \sqrt{25-16}}{4}$$

$$= \frac{-5 \pm 3}{4}$$

$$\Rightarrow z = \frac{-1}{2} \text{ and } z = -2$$

Singularities are $z = \frac{-1}{2}$ and $z = -2$.

$C; |z|=1$

When $z = \frac{-1}{2}$, $|z| = |\frac{-1}{2}| = \frac{1}{2} < 1$ lies inside C.

When $z = -2$, $|z| = |-2| = 2 > 1$ lies outside C.

Also $z = \frac{-1}{2}$ is a simple pole.

$$\therefore \operatorname*{Res}_{z=\frac{-1}{2}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-\frac{1}{2}}$$

$g(z) = 1$

$h(z) = 2z^2+5z+2$

$$= \left.\frac{1}{4z+5}\right|_{z=\frac{-1}{2}}$$

$$= \frac{1}{3}$$

$\therefore$ By Cauchy's Residue theorem (CRT),

$$\oint_C f(z)\,dz = 2\pi i\left[\text{Sum of residues at singular points inside } C.\right].$$

$$= 2\pi i \left[\operatorname*{Res}_{z=\frac{-1}{2}} f(z)\right]$$

$$= 2\pi i \left[\frac{1}{3}\right] = \frac{2\pi i}{3}$$

$$\therefore (1) \Rightarrow \int_0^{2\pi} \frac{1}{5+4\cos\theta} = \frac{1}{i} \times \frac{2\pi i}{3}$$

$$= \frac{2\pi}{3}$$

Q) Evaluate $\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta$ using contour integration.

Soln:

Let $z = e^{i\theta}$; $|z| = 1$

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \int_{C;|z|=1} \frac{1}{2+\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{1}{\frac{z^2+4z+1}{2z}} \frac{dz}{z}$$

$$= \frac{2}{i}\int_C \frac{1}{z^2+4z+1}\, dz \quad \text{--- (1)}$$

To find singularities,

$$z^2+4z+1 = 0 \Rightarrow z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

Singularities are $z = -2+\sqrt{3}$ and $z = -2-\sqrt{3}$.

When $z = -2+\sqrt{3}$; $|z| = |-2+\sqrt{3}| = \;\; < 1$, lies inside c

when $z = -2-\sqrt{3}$; $|z| = |-2-\sqrt{3}| = |2-\sqrt{3}| > 1$ lies outside c.

and $z = -2+\sqrt{3}$ is simple pole.

Now $\underset{z=-2+\sqrt{3}}{\text{Res}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=-2+\sqrt{3}}$

$$= \left. \frac{1}{2z+4} \right|_{z=-2+\sqrt{3}} \qquad \text{where } g(z) = 1$$

$$h(z) = z^2 + 4z + 1$$

$$= \frac{1}{-4+2\sqrt{3}+4} = \frac{1}{2\sqrt{3}}$$

$\therefore$ By CRT $\int_C f(z)\,dz = 2\pi i \left[\text{Sum of residues at singularity inside } c \right]$

$$= 2\pi i \left[\operatorname{Res}_{z=-2+\sqrt{3}} f(z) \right]$$

$$= 2\pi i \left[\frac{1}{2\sqrt{3}} \right]$$

$$= \frac{\pi i}{\sqrt{3}}$$

$\therefore$ ① $\Rightarrow$ required integral,

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\,d\theta = \frac{2}{i} \times \left[\frac{\pi i}{\sqrt{3}} \right]$$

$$= \frac{2\pi}{\sqrt{3}}$$

Q) Evaluate $\int_0^{\pi} \frac{1}{a+b\cos\theta} d\theta$, $a>b>0$ using contour integration.

Soln:

put, $z = e^{i\theta}$, $C: |z|=1$

$d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$

$$\therefore \int_0^{2\pi} \frac{1}{a+b\cos\theta} d\theta = \int_{C;|z|=1} \frac{1}{a+b\frac{(z^2+1)}{2z}} \frac{dz}{iz}$$

$$= \frac{1}{i} \int_C \frac{1}{\frac{bz^2+2az+b}{2z}} \frac{dz}{z}$$

$$= \frac{2}{i} \int_C \frac{1}{bz^2+2az+b} dz. \quad \text{--- (1)}$$

Now $bz^2+2az+b = 0 \Rightarrow z = \frac{-2a \pm \sqrt{(2a)^2-4b^2}}{2b}.$

$$\Rightarrow z = \frac{-2a \pm 2\sqrt{a^2-b^2}}{2b}$$

$$\Rightarrow z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$\therefore$ The singularities are,

$$z = \frac{-a+\sqrt{a^2-b^2}}{b} \quad \text{and} \quad z = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Here $z = \frac{-a + \sqrt{a^2 - b^2}}{b}$ lies inside C and

$z = \frac{-a - \sqrt{a^2 - b^2}}{b}$ lies outside C.

Also $z = \frac{-a + \sqrt{a^2 - b^2}}{b}$ is a simple pole.

$\therefore$ $\operatorname{Res}_{z = \frac{-a+\sqrt{a^2-b^2}}{b}} f(z) = \left. \frac{g(z)}{h'(z)} \right|_{z = \frac{-a + \sqrt{a^2 - b^2}}{b}}$

$$= \left. \frac{1}{2bz + 2a} \right|_{z = \frac{-a + \sqrt{a^2 - b^2}}{b}}$$

$$= \frac{1}{2b\left[\frac{-a + \sqrt{a^2 - b^2}}{b}\right] + 2a}$$

$$= \frac{1}{-2a + 2\sqrt{a^2 - b^2} + 2a} = \frac{1}{2\sqrt{a^2 - b^2}}$$

$\therefore$ By CRT, $\oint_C f(z)\, dz = 2\pi i \left[\text{sum of residues at singular points inside C} \right]$

$$= 2\pi i \left[\frac{1}{2\sqrt{a^2 - b^2}} \right] = \frac{\pi i}{\sqrt{a^2 - b^2}}$$

$\therefore$ (1) $\Rightarrow$ $\oint_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta = \frac{2}{i} \times \frac{\pi i}{\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ //.

2.2. Integrals of the form $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$

where $f(x)$ and $g(x)$ are polynomials such that no ~~zeros~~ poles ~~on~~ lies on the real axis.

Then $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx = \int_C \frac{f(z)}{g(z)} dz$ which

can be evaluated by using Residue theorem where c is the upper semi circle.

also $\int_{0}^{\infty} \frac{f(x)}{g(x)} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$

Q) Evaluate $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$ using contour integration.

Soln:

~~g(z)~~ Integrals is of the form $\int \frac{f(x)}{g(x)} = \int \frac{f(z)}{g(z)}$

~~where $g(z) = x^4 + 10x^2$~~

$g(x) = x^4 + 10x^2 + 9$

$\therefore g(z) = z^4 + 10z^2 + 9$

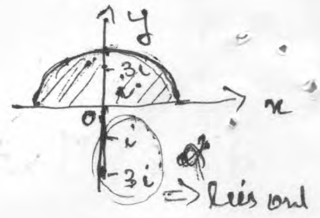
put $g(z) = z^4 + 10z^2 + 9 = 0$

$\Rightarrow (z^2)^2 + 10(z^2) + 9 = 0$

$\Rightarrow z^2 = \frac{-10 \pm \sqrt{100 - 36}}{2} = \frac{-10 \pm 8}{2} = -1, -9$

$\Rightarrow \quad z^2 = -1 \quad \text{and} \quad z^2 = -9$

$\Rightarrow \quad z = \pm i \quad \text{and} \quad z = \pm 3i$



Here $z = -i, -3i$ lies outside the upper semi circle c.

$z = i, +3i$ lies inside upper semi circle.

Also $z = i$ and $z = 3i$ are simple poles.

∴ By Cauchy's Residue theorem,

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\, dx = \int_C \frac{f(z)}{g(z)}\, dz$$

$$= 2\pi i \left[\text{Sum of residues at singular points within c.} \right].$$

Now $\operatorname{Res}_{z=i} f(z) = \left. \dfrac{g(z)}{h'(z)} \right|_{z=i}$

$$= \left. \frac{z^2 - z + 2}{4z^3 + 20z} \right|_{z=i}$$

where $g(z) = z^2 - z + 2$

$h(z) = z^4 + 10z^2 + 9$

$$= \frac{-1 - i + 2}{-4i + 20i} = \frac{1 - i}{16i}$$

Now $\operatorname{Res}_{z=3i} f(z) = \left. \dfrac{g(z)}{h'(z)} \right|_{z=3i}$

$$= \left. \frac{z^2 - z + 2}{4z^3 + 20z} \right|_{z=3i} = \frac{-9 - 3i + 2}{-108i + 60i}$$

$$= \frac{-3i-7}{-48i} = \frac{3i+7}{48i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{f(x)}{g(x)}\,dx = 2\pi i\left[\frac{1-i}{16i} + \frac{3i+7}{48i}\right]$$

$$= 2\pi i\left[\frac{10}{48i}\right] = \frac{5\pi}{12}$$

Q) Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

Soln:

Let $g(z) = z^4+1$

$g(z)=0 \Rightarrow z^4+1=0 \Rightarrow z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$

Here $z = e^{i5\pi/4}$ and $z = e^{i7\pi/4}$ lies outside the uppersemi circle C. [$\because$ 3rd quadrant & 4th quadr.]

~~Bec~~ And $z = e^{i\pi/4}$ and $z = e^{i3\pi/4}$ lies inside C.

Both are simple poles.

Now $\operatorname{Res}_{z=e^{i\pi/4}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=e^{i\pi/4}}$ [$\because g(z)=1$, $h(z) = z^4+1$]

$$= \left.\frac{1}{4z^3}\right|_{z=e^{i\pi/4}}$$

$$= \frac{1}{4\left(e^{i\pi/4}\right)^3} = \frac{1}{4} e^{-3i\pi/4}$$

$$= \frac{1}{4}\left(\cos 3\pi/4 - i \sin 3\pi/4\right)$$

$$= \frac{1}{4}\left(\frac{-1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right)$$

Now $\underset{z=e^{i3\pi/4}}{\text{Res}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=e^{i3\pi/4}} = \left.\frac{1}{4z^3}\right|_{z=e^{i3\pi/4}}$

$$= \frac{1}{4\cdot\left(e^{i3\pi/4}\right)^3} = \frac{1}{4} e^{-i9\pi/4}$$

$$= \frac{1}{4}\left[\cos(9\pi/4) - i \sin(9\pi/4)\right]$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right]$$

$\therefore$ By CRT, $\displaystyle\int \frac{f(z)}{g(z)} dz = 2\pi i \left[\begin{array}{l}\text{Sum of residues at}\\ \text{singular point inside } c\end{array}\right]$

$$= 2\pi i \left[\frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]$$

$$= 2\pi i \left[-\frac{1}{4}\cdot\frac{1}{\sqrt{2}} - i\frac{1}{4}\cdot\frac{1}{\sqrt{2}} + \frac{1}{4}\cdot\frac{1}{\sqrt{2}} - i\frac{1}{4}\cdot\frac{1}{\sqrt{2}}\right]$$

$$= 2\pi i \left[\frac{-2i}{4\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx = \frac{\pi}{\sqrt{2}}$$

Q) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

Soln:

The singularities are $g(z) = 0$

$$\Rightarrow (z^2+a^2)(z^2+b^2)=0$$

$$\Rightarrow z^2=-a^2 \text{ and } z^2=-b^2$$

$$\Rightarrow z=\pm ia \text{ and } z=\pm ib.$$

The singular points $z=-ia$ and $z=-ib$ lies outside the upper semicircle C.

The singular points $z=ia$ and $z=ib$ lies inside the upper semi circle C.

Now $\underset{z=ia}{\text{Res}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=ia} = \left.\frac{z^2}{(z^2+a^2)\cdot 2z+(z^2+b^2)\cdot 2z}\right|_{z=ia}$

$$= \frac{-a^2}{0+(-a^2+b^2)\cdot 2ia} = \frac{-a}{2i(-a^2+b^2)}$$

$$= \frac{-a}{-2i(a^2-b^2)} = \frac{a}{2i(a^2-b^2)}$$

$\underset{z=ib}{\text{Res}} f(z) = \left.\frac{g(z)}{h'(z)}\right|_{z=ib} = \left.\frac{z^2}{(z^2+a^2)\cdot 2z+(z^2+b^2)\cdot 2z}\right|_{z=ib}$

$$= \frac{-b^2}{(-b^2+a^2)2ib+0} = \frac{-b}{2i(-b^2+a^2)} = \frac{-b}{2i(a^2-b^2)}$$

$$= \frac{-b}{2i(a^2-b^2)} \quad //$$

$\therefore$ By CRT, $\int_c \frac{f(z)}{g(z)}\,dz = 2\pi i$ [sum of residues at singular point inside c]

$$= 2\pi i \left[\frac{a}{2i(a^2-b^2)} + \frac{-b}{2i(a^2-b^2)}\right]$$

$$= 2\pi i \left[\frac{(a-b)}{2i(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i}\left[\frac{(a-b)}{(a+b)(a-b)}\right]$$

$$= \pi\left(\frac{1}{a+b}\right)$$

$$= \frac{\pi}{a+b}$$

21/10/19

MODULE - 5

LINEAR SYSTEM OF EQUATIONS

Matrices: A Matrix is a rectangular array of numbers or functions which enclosed in a bracket.

eg) $A = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Elementary Row operations of Matrix

- → Interchange of 2 rows
- → Addition of a constant multiple of one row to another row.
- → Multiplication of a row by non-zero constant

eg) $A = \begin{bmatrix} 2 & 6 & 4 \\ 4 & 5 & 8 \\ 9 & 1 & 3 \end{bmatrix}$

Using matrix operations $R_2 \Rightarrow R_2 - 2R_1$

$\Rightarrow \begin{bmatrix} 2 & 6 & 4 \\ 0 & -7 & 0 \\ 9 & 1 & 3 \end{bmatrix}$ $R_3 \Rightarrow R_3 - \frac{9}{2}R_1$

$4-4 = 0$

$1 - \frac{9}{2}\times 6 = -26$, $5-12 = -7$

$8-8$

$3 - \frac{9}{2}\times 4 = -15$

$\Rightarrow \begin{bmatrix} 2 & 6 & 4 \\ 0 & -7 & 0 \\ 0 & -26 & -15 \end{bmatrix}$

A linear system of 'm' equations and 'n' unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

If all b_j's are ($j = 1, 2, \ldots, n$) ~~are~~ zero, the equation ① is called a homogeneous system of equations.

If atleast one b_j is not zero, then eqn ① is called non-homogenous system of equations.

A soln of eqn ① is a set of numbers $x_1, x_2, \ldots, x_n$ that satisfies all the m equations.

Eqn ① can be written as $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$ is called augmented matrix of system ①

Gauss Elimination method [Back substitution]

Q) Solve $2x_1 + 5x_2 = 2$, $-4x_1 + 3x_2 = -30$ using Gauss Elimination method.

$AX = B$

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

The augmented matrix, $\tilde{A} = \begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}$

Using Row reduced operations.

$R_2 \Rightarrow R_2 + 2R_1$

3 + 10
-30 + 4

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

$R_2 \Rightarrow R_2/13$

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 1 & -2 \end{bmatrix}$$

$2x_1 + 5x_2 = 2$

$x_2 = -2$

$2x_1 - 10 = 2$

$2x_1 = 12$

$x_1 = 6$

Solution of System of equations $\Rightarrow x_1 = 6,\ x_2 = -2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

Q) Solve $x + y + z = 3$

$x + 2y + 3z = 4$

$x + 4y + 9z = 6$

$AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix}$$

Using Row reduced opeeations

$$R_3 \Rightarrow R_3 - R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & 3 & 8 & 3 \end{bmatrix}$$

$x = 2$
$y = 1$
$z = 0$

$$R_2 \Rightarrow R_2 - R_1 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{bmatrix}$$

$$R_3 \Rightarrow R_3 - 3R_2 \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$8 - 6$

$x + y + z = 3 \quad - (1)$

$y + 2z = 1 \quad - (2)$

$2z = 0 \quad - (3)$

$z = 0$

Sub in (2),

$y = 1$

Sub in (3),

$x + 1 = 3$

$x = 2$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Q) By Gauss Elimination method, Solve

$x_1 - x_2 + x_3 = 0$

$-x_1 + x_2 - x_3 = 0$

$10x_2 + 25x_3 = 90$

$20x_1 + 10x_2 = 80$

Ans $AX = B$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$R_3 \to R_3/10$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 2 & 1 & 0 & 8 \end{bmatrix}$$

$R_2 \to R_2/5$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 18 \\ 2 & 1 & 0 & 8 \end{bmatrix}$$

$x_1 - x_2 + x_3 = 0$ — (1)

$2x_2 + 5x_3 = 18$ — (2)

$2x_1 + x_2 = 8$ — (3)

Solving (1) and (3)

$3x_1 + x_3 = 8 \times 5$

(3) $\Rightarrow 2x_2 + 5x_3 = 18$

$15x_1 + 5x_3 = 40$ — (6)

$(-)\ 2x_2 + 5x_3 = 18$ — (7)

$15x_1 - 2x_2 = 22$ — (8)

$40 - 18 = 22$

$R_4 \Rightarrow R_4 - 20R_1 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$

$R_2 \to R_2 + R_1 \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$

$R_4 \to R_4 - 3R_3$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \end{bmatrix}$$

$\Rightarrow -95x_3 = -190 \Rightarrow x_3 = \frac{-190}{-95} = 2$

$10x_2 + 25x_3 = 90$

$10x_2 + 50 = 90$

$x_2 = \frac{40}{10} = 4$

$x_1 - x_2 + x_3 = 0$

$x_1 - 4 + 2 = 0 \Rightarrow x_1 = 2$

Note

The linear system is said to be consistent if it has a solution and inconsistent if it has no solution.

If the system is consistent, it has finitely or infinitely many solutions.

Rank of a matrix

To find the rank of a matrix, reduce the matrix to row echelon form and the no. of non-zero rows determines the rank.

The echelon form, The starting non-zero element in each row must be 1 and the column containing that 1 should be 0's.

Q) Find the rank of the matrix, $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} 1 & -1 & -2 & -1 \\ 2 & 3 & -1 & 1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -1 \end{bmatrix}$

Scanned with CamScanner

$R_2 \Rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 3 \\ 0 & 4 & 9 & 1 \\ 6 & 3 & 0 & -1 \end{bmatrix}$$

$R_4 \Rightarrow R_4 - 6R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 5 & 3 & 3 \\ 0 & 4 & 9 & 1 \\ 0 & 9 & 12 & 5 \end{bmatrix}$$

~~$R_3 \Rightarrow R_3 + 4R_1$~~ $R_2 \Rightarrow R_2/5$, $R_3 \Rightarrow R_3/9$, $R_4 \Rightarrow R_4/5$

$$\begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 3/5 & 3/5 \\ 0 & 4/9 & 1 & 1/9 \\ 0 & 9/5 & 12/5 & 1 \end{bmatrix}$$

~~$R_3 + \frac{4}{9}R_1$~~ $R_3 + \frac{4}{9}R_1 \Rightarrow R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 3/5 & 3/5 \\ 0 & 4/9 & 1 & 1/9 \\ 0 & 9/5 & 12/5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 3/5 & 3/5 \\ 0 & 0 & 1/9 & 1/9 \\ 0 & 9/5 & 12/5 & 1 \end{bmatrix}$$

$(R_3 \Rightarrow R_3 \times 9)$

$$\begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 3/5 & 3/5 \\ 0 & 0 & 1 & 1 \\ 0 & 9/5 & 12/5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & 3/5 & 3/5 \\ 0 & 0 & 1 & 1 \\ 9/5 & 0 & & \end{bmatrix}$$

$R_4 \Rightarrow R_4 + \frac{9}{5}$

<u>Imp</u> <u>To check the given system of equations is consistent or not.</u>

If rank of AB not equal to rank of A:

→ The given system of equation is inconsistent and has no solution

rank (AB) ≠ rank (A) where [AB] is the augmented matrix.

→ If rank (AB) = rank (A) = no. of variables

⇒ consistent and has a unique solution

→ If rank (AB) = rank (A) < no. of variables

⇒ consistent and has infinite no. of solutions.

Q) Solve by Gauss elimination: $x_1 - x_2 + x_3 = 0$, $-x_1 + x_2 - x_3 = 0$, $10x_2 + 25x_3 = 90$, $20x_1 + 10x_2 = 80$. Check whether the system is consistent or not.

<u>ans</u>

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix} \quad R_2 \Rightarrow R_1 + R_2, \; R_4 \Rightarrow R_4 - 20R_1$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \end{bmatrix} \quad R_4 \Rightarrow R_4 - 3R_3$$

$-20 - 75 = -95$, $80 - 270$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 25/10 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_4 \Rightarrow -R_4/95$, $R_3 \Rightarrow R_3/10$

$x_1 - x_2 + x_3 = 0$

$x_2 + 25/10\, x_3 = 9$

$\underline{x_3 = 2}$

$x_2 + \frac{25}{10_5} x_2 = 9$

$\underline{x_2 = 4}$

$x_1 - 4 + 2 = 0$

$\underline{x_1 = 2}$

$\Rightarrow$ consistent and has a unique solution

Q) By Gauss Elimination method $3x + 2y + 2z = 1$, $x + 2y = 4$, $10y + 3z = -2$, $2x - 3y - z = 5$

$$AB = \begin{bmatrix} 3 & 2 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$\tilde{AB} = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$ Now reducing using operations

$R_1 \leftrightarrow R_2$

$$\tilde{AB} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$$\tilde{AB} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$ $R_2 \to R_2 - 3R_1$, $R_4 \to R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$ $R_2 \to R_2/-3$

$$= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}$$ $R_3 \to R_3 - 10R_2$, $R_4 \to R_4 + 7R_2$

$$= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -17/3 & 68/3 \end{bmatrix}$$ $R_3 \to \frac{3}{29} R_3$

$$= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -2/3 & 11/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_4 \to R_4 + \frac{17}{3} R_3$

$x_1 + 2x_2 = 4$

$x_2 - 2/3 = 11/3 \Rightarrow x_2 = \underline{13/3}$

$x_1 + 26/3 = 4$

$x_1 = \underline{-14/3}$

$x_3 = -4$

$4 - \frac{26}{3}$, $\frac{12 - 26}{3}$, -14

consistent and unique solution

Q) $y + z - 2w = 0$, $2x - 3y - 3z + 6w = 2$, $4x + y + z - 2w = 4$

$$\begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$R_1 \Rightarrow R_1/2$

$$\begin{bmatrix} 1 & -3/2 & -3/2 & 3 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & -3/2 & -3/2 & 3 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 4 & 0 & 0 & 0 & -4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & -3/2 & -3/2 & 3 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & -6 & -6 & +12 & 0 \end{bmatrix}$$

$0 \quad 12/2$

$0 \quad -12$

$R_3 \rightarrow R_3/6$

$$\begin{bmatrix} 1 & -3/2 & -3/2 & 3 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & -3/2 & -3/2 & 3 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank (AB) = 2, rank (A) = 2,

No. of variables = 4

$2 < 4$

$\Rightarrow$ Consistent and infinitely many solutions

$x - 3/2\, y - 3/2\, z + 3w = 1$

$y + z = 2w$

Eg $y = 1,\ z = 1$, $w = 1$

$y = 2,\ z = 2$, $w = 2$

$y = r,\ z = t,$

$r + t = 2w \qquad w = \frac{r+t}{2}$

$x - \frac{3}{2}r - \frac{3}{2}t + \frac{3}{2}r + \frac{3}{2}t = 1$

$x = 1$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ r \\ t \\ \frac{r+t}{2} \end{bmatrix}$$

31/10/19

Q) Find the value of μ for which the system of equation $x+y+z=1$, $x+2y+3z=\mu$, $x+5y+9z=\mu^2$ will be consistent. For each value of μ, find the solution of system.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & \mu \\ 1 & 5 & 9 & \mu^2 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1$
$R_3 \Rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 4 & 8 & \mu^2-1 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 0 & 0 & \mu^2-4\mu+3 \end{bmatrix}$$

μ^2-1
$-(4\mu-4)$

$R(AB) = R(A) = \mu^2 - 4\mu + 3 = 0$

$\mu = 1, 3$

The system will be consistent rank $[A]$ = rank $[B]$

This is possible only if $\mu^2 - 4\mu + 3 = 0$

$\mu = 1$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow y + 2z = 0$$

put $z = t$,

$y + 2t = 0$

$y = -2t$

$x + y + z = 1$

$x - 2t + t = 1$

$x - t = 1$

$x = 1+t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ -2t \\ t \end{bmatrix}$$

$\mu = 3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$y + 2z = 2$

$z = t$

$y = 2 - 2t$

$x + y + z = 1$

$x + 2 - 2t + t = 1$

$x + 2 - t = 1$

$x = -1 + t$

$x = t - 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t-1 \\ 2-2t \\ t \end{bmatrix}$$

Q) Find the value of λ and μ for which the system $2x + 3y + 5z = 9$; $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$

The system has (i) one soln (ii) Infinite solns (iii) one parameter family of soln.

$$\begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$R_1 \to R_1/$

$$\begin{bmatrix} 1 & 3/2 & 5/2 & 9/2 \\ 7 & 3 & -2 & 8 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$$

$R_2 \to R_2 - 7R_1$
$$\begin{bmatrix} 1 & 3/2 & 5/2 & 9/2 \\ 0 & -15/2 & -39/2 & -47/2 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$$

$8 - \frac{21}{2}$, $5 - \frac{21}{2}$, $-2 - 35/2$, $-4 - \frac{35}{2}$, $8 - \frac{63}{2}$, $\frac{16-63}{2}$

(i) Possible only if $\lambda - 5 = 0$ and $\mu - 9 \neq 0$

(ii) Possible only if $\lambda \neq 5$ and $\mu \neq 9$

(iii) The system has one parameter family of soln if $R(AB) = R(A) <$ no. of variables

i.e, $2 < 3$

This is possible only if $\lambda = 5$ and $\mu = 9$ ($\lambda - 5 = 0$, $\mu - 9 = 0$)

$-15y - 39z = -47$

put $z = t$,

$-15y - 39t = -47$

$-15y = -47 + 39t$

$y = \frac{-47 + 39t}{-15}$

$x + 3/2\, y + 5/2\, z = 9/2$

$2x + 3y + 5z = 9$

$2x + 3\left(\frac{-47 + 39t}{-15}\right) + 5t = 9$

$2x + 3\left(\frac{-47 + 39t}{-15}\right) + 5t = 9$

$2x + \left(\frac{-47 + 39t}{-5}\right) + 5t = 9$

$2x = 9 + \left(\frac{-47 + 39t}{5}\right) - 5t$

$2x = \frac{45 - 47 + 39t - 25t}{5}$

$x = \frac{-\frac{2}{5} + \frac{14t}{5}}{2}$

$= -\frac{2}{10} + \frac{14}{10}t$

$x = \frac{-1}{5} + \frac{7}{5}t$

1/11/2019

Linear Combination of Vectors

Let $a_1, a_2, \dots, a_m$ are vectors in R^n and $\alpha_1, \alpha_2, \dots, \alpha_m$ are scalars, then $\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_m a_m$ is called Linear combination of vectors

Q) Express the vector $(1, -2, 5)$ as a linear combination of vectors $(1,1,1)$, $(1,2,3)$ and $(2,-1,1)$

$(1, -2, 5) = \alpha_1(1,1,1) + \alpha_2(1,2,3) + \alpha_3(2,-1,1)$

$1 = \alpha_1 + \alpha_2 + 2\alpha_3$ — ①

$-2 = \alpha_1 + 2\alpha_2 - \alpha_3$ — ②

$5 = \alpha_1 + 3\alpha_2 + \alpha_3$ — ③

Equating ① and ②

$\alpha_1 + \alpha_2 + 2\alpha_3 = 1$

(−) $\alpha_1 + 2\alpha_2 + \alpha_3 = -2$

$0 - \alpha_2 + 3\alpha_3 = 3$ — ④

① and ③

$\alpha_1 + \alpha_2 + 2\alpha_3 = 1$

(−) $\alpha_1 + 3\alpha_2 + \alpha_3 = 5$

$-2\alpha_2 + \alpha_3 = -4$ — ⑤

④ and ⑤

$-\alpha_2 + 3\alpha_3 = 3 \quad \times 2$

$-2\alpha_2 + \alpha_3 = -4$

$-2\alpha_2 + 6\alpha_3 = 6$

(−) $-2\alpha_2 + \alpha_3 = -4$

$5\alpha_3 = 10$

$\alpha_3 = 2$

$-\alpha_2 + 3\alpha_3 = 3$

$-\alpha_2 + 6 = 3$

$-\alpha_2 = -3$

$\alpha_2 = 3$

$\alpha_1 + \alpha_2 + 2\alpha_3 = 1$

$\alpha_1 + 3 + 4 = 1$

$\alpha_1 = 1 - 7$

$\alpha_1 = -6$

Linearly independent and Linearly dependent vectors

The vectors are <u>linearly independent</u> if

$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$ and linearly dependent:

$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0$

$\Rightarrow$ atleast one $\alpha_i \neq 0$

Note

The vectors are linearly independent if $R(A) =$ No. of vectors or determinant $A \neq 0 \Rightarrow |A| \neq 0$

The vectors are linearly dependent if $R(A) <$ No. of vectors or determinant $A = 0 \Rightarrow |A| = 0$

Q) Show that the vectors $(1, 1, 1)$, $(1, 2, 3)$, $(2, -1, 1)$ are linearly independent.

Soln: vectors are linearly independent if $R(A) =$ No. of vectors

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

No. of vectors = 3

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -2 \end{bmatrix}$$

$R(A) = 3$ = No. of variables.

$\Rightarrow$ Linearly independent

OR

$$|A| = 1(2+3) - 1(1-6) + 1(-1-4)$$
$$= 5 + 5 - 5$$
$$= 5 \neq 0$$

$\Rightarrow$ Linearly independent.

Q) Test whether the vectors $(1, 1, 1)$, $(1, -1, 1)$, $(3, -1, 3)$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & -1 & 3 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1$
$R_3 \Rightarrow R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R(A) = 2$

$R(A) \neq$ No. of vectors

$R(A) <$ No. of vectors

$\Rightarrow$ Linearly dependent

Q) Verify whether the vectors $(1, 2, 1, 2)$, $(3, 1, -2, 1)$, $(4, -3, -1, 3)$, $(2, 4, 2, 4)$ are linearly independent or not

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix} \quad R_4 \Rightarrow R_4 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 4 & -3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \Rightarrow R_2/-5$, $R_3 \Rightarrow R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ \ldots \end{bmatrix}$$

$R(A) = 3$

No. of vectors $= 4$

$R(A) <$ No. of vectors

$\Rightarrow$ Linearly dependent

Basis of row space, column space and null space

Q) Reduce the given matrix to echelon form. The non-zero rows determines the basis of row space. Reduce the matrix A^T to echelon form and the non-zero rows determines the basis of column space. The solution set of homogeneous system $(AX = 0)$ determines the basis of the Null space and the dimension of null space is called nullity of A.

Q) Find row space, column space and null space of $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & 3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$

Basis of row space

Set of non-zero rows

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

$R_4 \Rightarrow R_4 - 3R_1$, $R_2 \rightarrow R_2 + 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix}$$

$-5+4$
$1+$
$-1+4$
$-8+10$
$6-6$
$-7-6$
$2-15$

$R_4 \Rightarrow R_4 / -13$

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 3R_2$

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$4-9$
$1-6$

$R_3 \Rightarrow R_3 / -5$

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_4 \Rightarrow R_4 - R_3$

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of row space = non-zero rows in echelon form

$$A = \{(1, 2, 0, 2, 5), (0, -1, 1, 3, 2), (0, 0, 0, 1, 1)\}$$

Dimension of row space = 3

Basis of column space

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -5 & -3 & 6 \\ 0 & 1 & 3 & 0 \\ 2 & -1 & 4 & -7 \\ 5 & -8 & 1 & 2 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_1$, $R_4 \Rightarrow R_4 - 2R_1$, $R_5 \Rightarrow R_5 - 5R_1$

$$= \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \end{bmatrix}$$

-1+4
-5+4
6-6
-1+4
4-0
-7-6
-8+10
1-
2-15

$R_2 \Rightarrow -R_2$

$$= \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \end{bmatrix}$$

$R_5 \Rightarrow R_5 - R_4$

$$= \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & -1 & -3 & 0 \end{bmatrix}$$

2-

$R_5 \Rightarrow R_5 + R_2$

$$= \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of column space = non-zero rows in echelon form.

$$A = \{(1, -2, 0, 3), (0, 1, 3, 0), (0, 3, 4, -13)\}$$

Dimension of column space = 3

Null space

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$x_4 + x_5 = 0$ — ①

$-x_2 + x_3 + 3x_4 + 2x_5 = 0$ — ②

$x_1 + 2x_2 + 2x_4 + 5x_5 = 0$ — ③

$x_4 = t$

$x_4 = -x_5 = -t$

$-x_2 + x_3 + 3t - 2t = 0$

$-x_2 + x_3 = t$ — ④

Scanned with CamScanner

$x_1 + 2x_2 + 2t - 5t = 0$

$x_1 + 2x_2 = 3t$ — (5)

(4) and (5)

$0 - x_2 + x_3 = -t$

$x_1 + 2x_2 + 0 = 3t$

$x_1 + x_2 + x_3 = 2t$

$x_3 = s,$

$x_1 + x_2 = 2t - s$

$x_1 = -2s - t$

$x_2 = s - t$

$$\begin{bmatrix} -2s - t \\ s - t \\ s \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ -t \\ 0 \\ -t \\ t \end{bmatrix} =$$

$$X = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Before null space

(Q) find the basis of null space.

Solve the homogeneous system of equations $AX = 0$, we have row reduced echelon form of matrix,

Rank (A) = 3

Basis of null space = {(-2, 1, 0, 0, 0), (-1, -1, 0, -1, 1)}

Dimension of null space = 2

Basis of null space = solution set of homogeneous system $AX = 0$.

MODULE

Matrix Eigen Value Problems

Consider the vector equation,

$$Ax = \lambda x \quad \text{— (1)}$$

where A is a given square matrix, λ is an unknown constant and x is an unknown vector.
The MEV Problem is to find λ and x that satisfies (1).
The value of λ that satisfy (1) is called Eigen Value of A and corresponding value of x is called eigen vector of A.

How to find Eigen Value and Eigen Vector

Consider the vector equation,

$$Ax = \lambda x$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = \lambda x_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = \lambda x_2$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = \lambda x_n$$

ie $(a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$

$a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n = 0$

$a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n = 0$

In matrix notation, this can be written as.

$$[A-\lambda I]x = 0$$

where I is the identity matrix with same order of A.
The matrix $A-\lambda I$ is called characteristic matrix.
$D(\lambda) = |A-\lambda I|$ is called characteristic determinant.
The equation $|A-\lambda I| = 0$ is called characteristic equn: of A.
By expanding characteristic equn: we get a polynomial of degree n and that polynomial is called characteristic polynomial of A.

The eigen values of square matrix A are the roots of characteristic equn: of A.
Hence $n \times n$ matrix has atleast one eigen value and almost n eigen value.

Find Eigen Value and Eigen Vector

4) $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$[A-\lambda I] = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -5-\lambda & 2 \\ 2 & (-2-\lambda) \end{bmatrix}$

$|A-\lambda I| = 0$

$(-5-\lambda)(-2-\lambda) - 4$

$= 10 + 5\lambda + 2\lambda + \lambda^2 - 4$

$0 = \lambda^2 + 7\lambda + 6$

$\lambda = -1, -6$ are the eigen values.

Now consider $[A-\lambda I]x$

$= \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$[A-\lambda I]x = 0$

$(-5-\lambda)x_1 + 2x_2 = 0$

$2x_1 - (2+\lambda)x_2 = 0$

$\lambda = -1$

$-4x_1 + 2x_2 = 0$

$2x_1 - x_2 = 0$

$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$

$\begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_2 \rightarrow 2R_2 + R_1$

$-4x_1 + 2x_2 = 0$

$x_2 - 2x_1 = 0$

$x_2 = 2x_1$

$n = 2$

$n > R$

$(n - R) = 1$

Put $x_1 = 1$ $\quad X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x_2 = 2$

$\lambda = -6$

$2x_1 + 4x_2 = 0$

$x_1 + 2x_2 = 0$ — (1)

$x_1 + 2x_2 = 0$ — (2)

$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ $R_2 \to R_2 - R_1$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$R_A = 1 \quad R_{\tilde{A}} = 1$

$n = 2$

$(n - R) = 1$

Put $x_2 = 1$

$x_1 = -2$

$X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

$[A - \lambda I] = \begin{bmatrix} -\lambda & 4 \\ -4 & -\lambda \end{bmatrix}$

$|A - \lambda I| = 0$

$\lambda^2 + 16 = 0$

$\lambda = \pm 4i$

$\lambda = +4i, -4i$ are eigen values

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -\lambda & 4 \\ -4 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-\lambda x_1 + 4x_2 = 0$$
$$-4x_1 - \lambda x_2 = 0$$

$$-\lambda x_1 + 4x_2 = 0$$
$$-4x_1 - \lambda x_2 = 0$$

$$\lambda = 4i$$

$$-4i x_1 + 4x_2 = 0$$
$$ix_1 + x_2 = 0 \longrightarrow (1)$$
$$-4x_1 - 4ix_2 = 0$$
$$x_1 + ix_2 = 0 \Longrightarrow (2)$$

$$\begin{bmatrix} -i & 1 & 0 \\ 1 & i & 0 \end{bmatrix} \quad R_2 \rightarrow iR_2 + R_1$$

$$\begin{bmatrix} -i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$
$$x_2 = ix_1$$

$$x_1 = 1$$
$$x_2 = i$$
$$X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda = -4i$$

$$4ix + 4x_2 = 0$$
$$ix_1 + x_2 = 0 \quad - (1)$$
$$-4x_1 + 4ix_2 = 0$$
$$-x_1 + ix_2 = 0 \quad - (2)$$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \end{bmatrix} \quad R_2 \rightarrow iR_2 + R_1$$

$$\begin{bmatrix} i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$ix_1 = -x_2 \qquad x_2 = 1 \qquad x_1 = \frac{-1}{i}$$

$$X = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

5) $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix}$$

$$4-\lambda \left((5-\lambda)(3-\lambda)\right] - 2\left[(6-2\lambda) - 2\left[10-2\lambda\right)\right]$$

$$4-\lambda\left[15-5\lambda-3\lambda+\lambda^2\right] - 12+4\lambda \quad -20+4\lambda$$

$$(4-\lambda)(\lambda^2-8\lambda+15) + (8\lambda-32)$$

$$(4-\lambda)(\lambda^2-8\lambda+15) - 8(4-\lambda)$$

$$(4-\lambda)(\lambda^2-8\lambda+7) = 0$$

$\lambda = 4$, $\lambda = 7, 1$ are eigen values..

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$(4-\lambda)x_1 + 2x_2 - 2x_3 = 0$$

$$2x_1 + (5-\lambda)x_2 = 0$$

$$-2x_1 + (3-\lambda)x_3 = 0$$

$\lambda = 4$

$$x_2 - x_3 = 0$$

$$2x_1 - x_2 = 0$$

$$-2x_1 - x_3 = 0$$

$$\begin{bmatrix} 0 & 2 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2 \longrightarrow \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$2x_1 + x_2 = 0$

$x_1 = 1 \quad x_2 = -2$

$x_2 - x_3 = 0$

$-2 - x_3 = 0 \quad \underline{x_3 = -2}$

$$X = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$\lambda = 1$

$3x_1 + 2x_2 - 2x_3 = 0$

$2x_1 + 4x_2 = 0$

$-2x_1 + 2x_3 = 0$

$$A_1 = \left[\begin{array}{ccc|c} 3 & 2 & -2 & 0 \\ 2 & 4 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right]$$

$R_2 \to 3R_2 - 2R_1 \Longrightarrow \begin{bmatrix} 3 & 2 & -2 & 0 \\ 0 & 8 & 4 & 0 \\ -2 & 0 & 2 & 0 \end{bmatrix}$

$R_3 \to R_3 - 2R_2 \implies \begin{bmatrix} 3 & 2 & -2 & | & 0 \\ 6 & 4 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

$R < n$

$(n-R) = 1$

$3x_1 + 2x_2 - 3x_3 = 0$

$4x_2 + 2x_3 = 0$

ie $4x_2 = -2x_3$

Put $x_2 = 1$ $\quad x_3 = -2$

$3x_1 + 2x_2 - 3x_3 = 0$

$3x_1 = -2 \times 1 + 3x - 2 = -2 - 6$

$x_1 = -8/3$

$X_2 = \begin{bmatrix} -8/3 \\ 1 \\ -2 \end{bmatrix}$

$\lambda = 7$

$-3x_1 + 2x_2 - 2x_3 = 0$

$2x_1 - 2x_2 + 0 = 0$

$-2x_1 - 4x_3 = 0$

$A_2 = \begin{bmatrix} -3 & 2 & -2 & 0 \\ 2 & -2 & 0 & 0 \\ -2 & 0 & -4 & 0 \end{bmatrix}$

SYMMETRIC AND SKEW SYMMETRIC MATRIX

A matrix A is called Symmetric if $A = A^T$ and A is is called Skew Symmetric if $A = -A^T$.

Consider the matrix $A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$

$$A^T = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$A = A^T$ ∴ matrix is Symmetric

Consider the matrix $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

$A = -A^T$

∴ A is skew Symmetric

Orthogonal Matrices

A matrix A is called orthogonal if $A^T = A^{-1}$. To find inverse of matrix A, if A is a 2x2 matrix, then,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then $A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Let A be a square matrix of order greater than 2. Then we can find inverse of A using row transformations.

For that consider the matrix.

$$\left[\, A \,\middle|\, I \,\right]$$

where I is an identity matrix of same order.

Then using row transformation convert matrix A into an identity matrix.

The new form of I will give A^{-1}.

ie $\left[\, I \,\middle|\, A^{-1} \,\right]$

Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right]$$

$R_2 \rightarrow -R_2$

$R_3 \rightarrow -R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

check whether A is symmetric, skew symmetric and orthogonal

$$A^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$A^{-1} = A^T$

∴ Given matrix is Orthogonal.

The set of all eigen values of A is called spectrum of A.

2) Check whether the matrix A is symmetric, skew symmetric or orthogonal. find the spectrum of A and find eigen values?

$$A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$= 1/5 \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{25} & 4/25 \\ -4/25 & 3/25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{25} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$A \neq A^T$

$A \neq -A^T$

$A^T = A^{-1}$

∴ it is orthogonal

$$[A-\lambda I] = \begin{bmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{bmatrix}$$

$$|A-\lambda I| = 0$$

$$(3-\lambda)^2 + 16 = 0$$

$$25 + \lambda^2 - 6\lambda = 0$$

$\lambda = 3+4i,\ 3-4i$ are the eigen values.

$\{3+4i, 3-4i\}$ = Spectrum

$$[A-\lambda I]x = 0$$

$$\begin{bmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$(3-\lambda)x_1 - 4x_2 = 0$$

$$4x_1 + (3-\lambda)x_2 = 0$$

Put $\lambda = 3-4i$

$$4i\,x_1 - 4x_2 = 0$$

$$i x_1 - x_2 = 0 \quad \text{——} (1)$$

$$4x_1 + 4i\,x_2 = 0$$

$$x_1 + i x_2 = 0$$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right]$$

$R_2 \rightarrow iR_2 - R_1$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$i x_1 - x_2 = 0$$

$$i x_1 = x_2$$

~~x_1~~ $x_1 = 1$

$x_2 = i$

$\therefore X = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigen Vector

Put $\lambda = 3+4i$

$-ix_1 - x_2 = 0$

$ix_1 + x_2 = 0$ — ①

$x_1 - ix_2 = 0$

$$\left[\begin{array}{cc|c} 4i & 4 & 0 \\ 4 & -4i & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 i - R_1$

$$\left[\begin{array}{cc|c} 4i & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$ix_1 = x_2$

$x_1 = 1$

$x_2 = -i$

$X = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Eigen Vectors are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

Diagonalisation of Matrix

A matrix A is called diagonalisable if there exist a matrix P such that $P^{-1}AP$ is a diagonal matrix.

Theorem

If a square matrix A has a basis of eigen vectors then $D = X^{-1}AX$ is a diagonal matrix with eigen values of A as the entries on the main diagonal. Here X is the matrix with the eigen vectors as column vectors.

1) Find an Eigen basis and diagonalise.

$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$4 + \lambda^2 - 4\lambda - 16 = 0$

$\lambda^2 - 4\lambda - 12 = 0$

$\lambda = 6, -2$ are eigen values

$(A - \lambda I)X = 0$

$(2-\lambda)x_1 + 4x_2 = 0$

$4x_1 + (2-\lambda)x_2 = 0$

<u>at $\lambda = -2$</u>

$4x_1 + 4x_2 = 0$

$4x_1 + 4x_2 = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = -x_1$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Put $\lambda = 6$

$-x_1 + x_2 = 0$

$x_1 - x_2 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_2$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$X^{-1} = \frac{1}{|A|}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$X^{-1}A = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}$$

1) $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

$|A-\lambda I| = 0$

$$\begin{bmatrix} -19-\lambda & 7 \\ -42 & 16-\lambda \end{bmatrix} = 0$$

$-304 + 19\lambda - 16\lambda + \lambda^2 + 294$

$0 = \lambda^2 + 3\lambda - 10$

$\lambda = -5, 2$

$(-19-\lambda)x_1 + 7x_2 = 0$

$-42x_1 + (16-\lambda)x_2 = 0$

$\lambda = 2$

2) $A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & -1 & 0 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$-1-\lambda\left(-2+\lambda-2\lambda+\lambda^2\right)+1(\lambda-2)=0$$

$$2-\lambda+2\lambda-\lambda^2+2\lambda-\lambda^2+2\lambda^2-\lambda^3+\lambda-2=0$$

$$4\lambda-\lambda^3=0$$

$$\lambda^3-4\lambda$$

$$\lambda(\lambda^2-4)=0$$

$$\lambda = 2, -2, 0$$

$$|A-\lambda I|X = 0$$

$$\begin{bmatrix} -1-\lambda & -1 & 0 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$(-1-\lambda)x_1 - x_2 = 0$$

$$-x_1 \cdot (-1-\lambda)x_2 = 0$$

$$(2-\lambda)x_3 = 0$$

$\lambda = -2$

$$x_1 - x_2 = 0$$

$$-x_1 + x_2 = 0$$

$$4x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$4x_3 = 0$

$x_3 = 0$

$x_1 - x_2 = 0$

$x_1 = x_2$

Put $x_1 = 1$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Put $\lambda = 2$

$$\begin{bmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$-3x_1 - x_2 = 0$ $\quad\quad$ $-3x_1 = x_2$

$-x_2 = 0$ $\quad\quad$ $x_1 = x_2 = 0$

Put $x_3 = 1$

$\therefore X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Put $\lambda = 0$

$-x_1 - x_2 = 0$

$-x_1 - x_2 = 0$

$2x_3 = 0$

$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 - R_1$$

$-x_1 - x_2 = 0$

$x_1 = x_2$

$$2x_3 = 0$$
$$x_3 = 0$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1 + 1/2\, R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$R_3 \to -1/2\, R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right]$$

$X^{-1}A$.

$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= [-2, 2, 0]$$

Similar Matrices

An $n \times n$ matrix $\hat{A}$ is called similar to an $n \times n$ matrix A if $\hat{A} = P^{-1}AP$ for some non-singular $n \times n$ matrix P. If $\hat{A}$ is similar to A, then $\hat{A}$ has the same Eigen values of A. If X is an eigen vector of A, then $Y = P^{-1}X$ is an eigen vector of $\hat{A}$.

1) Find $\hat{A}$ and show that, if y is an eigen vector of $\hat{A}$, then $x = PY$ is an eigen vector of A.

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -3 \\ -2 & -4 \end{bmatrix}$$

$$P^{-1}A = \frac{1}{-2}\begin{bmatrix} -15 & 5 \\ -22 & 4 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{-2}\begin{bmatrix} 15 & 5 \\ -22 & 4 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$$

$$|\hat{A} - \lambda I| = 0$$

$$\begin{vmatrix} -25-\lambda & 12 \\ -50 & 25-\lambda \end{vmatrix} = 0$$

$$(-25-\lambda)(25-\lambda) + 50 \times 12 = 0$$

$-625 + 25\lambda - 25\lambda + \lambda^2 + 600 = 0$

$$\lambda^2 - 25 = 0$$

$$\lambda = \pm 5$$

$\lambda = -5$

$$(\hat{A} - \lambda I)x = 0$$

$$(-25-\lambda)x_1 + 12x_2 = 0$$

$$-50x_1 + (25-\lambda)x_2 = 0$$

$$-20x_1 + 12x_2 = 0$$

$$-50x_1 + 30x_2 = 0$$

$$\left[\begin{array}{cc|c} -10 & 6 & 0 \\ -5 & 3 & 0 \end{array}\right]$$

$$x_2 \rightarrow 2x_2 - x_1$$

$$\left[\begin{array}{cc|c} -10 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-10x_1 + 6x_2 = 0$$

$$x_1 = \frac{6x_2}{10}$$

$$X = \begin{bmatrix} 1 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$-30x_1 + 12x_2 = 0$

$-50x_1 + 20x_2 = 0$

$$\begin{bmatrix} -30 & 12 & 0 \\ -50 & 20 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2/10$$

$$\begin{bmatrix} -\frac{30}{5} & 12 & 0 \\ & 2 & 0 \end{bmatrix}$$

$$R_2 \rightarrow 6R_2 - R_1$$

$$\begin{bmatrix} -30 & 12 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$30x_1 + 12x_2 = 0$$

$$X = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

2) $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}$

$\lambda = -5$

$[A - \lambda I]x = 0$

$(3-\lambda)x_1 + 4x_2 = 0$

$4x_1 + (-3-\lambda)x_2 = 0$

$8x_1 + 4x_2 = 0$

$2x_1 + x_2 = 0$ —— (1)

$4x_1 + 2x_2 = 0$

$2x_1 + x_2 = 0$ —— (2)

$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

$R_2 \rightarrow R_2 - R_1$

$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$2x_1 = -x_2$

$x_1 = 1$

$X = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 5$

$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$-2x_1 + 4x_2 = 0$

$4x_1 - 8x_2 = 0$

$\begin{bmatrix} -2 & 4 & 0 \\ 4 & -8 & 0 \end{bmatrix}$

$R_2 \rightarrow 2R_1 + R_2$

$\begin{bmatrix} -2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$-2x_1 + 4x_2 = 0$

$x_1 = 2x_2$

$X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Take $\lambda = 5$

y - eigenvector of $\hat{A}$

$$y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$X = Py$

$$= \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

at $\lambda = -5$

$$y = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$X = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

3) $A = \begin{bmatrix} -2 & 0 & 12 \\ -2 & 4 & 4 \\ 2 & 0 & 12 \end{bmatrix}$ $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 \Leftrightarrow R_1$$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 12 \\ -2 & 4 & 4 \\ -2 & 0 & 12 \end{bmatrix}$$

$$P^{-1}AP \begin{bmatrix} -2 & 4 & 4 \\ -2 & 0 & 12 \\ -2 & 0 & 12 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 4 & -2 & 4 \\ 0 & -2 & 12 \\ 0 & -2 & 12 \end{bmatrix}$$

$|A-\lambda I| = 0$

$$\begin{bmatrix} 4-\lambda & -2 & 4 \\ 0 & -2-\lambda & 12 \\ 0 & -2 & 12-\lambda \end{bmatrix}$$

$$4-\lambda \left[(-2-\lambda)(12-\lambda) + 24\right] = 0$$

$4-\lambda = 0$

$\lambda = 4$

$-24 + 2\lambda - 12\lambda + \lambda^2 + 24 = 0$

$-10\lambda + \lambda^2 = 0$

$\lambda(\lambda - 10) = 0$

$\lambda = 10, 0, 4$

at $\lambda = 0$

$$\begin{bmatrix} 4 & -2 & 4 \\ 0 & -2 & 12 \\ 0 & -2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$4x_1 - 2x_2 + 4x_3 = 0$

$2x_1 - x_2 + 2x_3 = 0$ —— (1)

$-2x_2 + 12x_3 = 0$

$-x_2 + 6x_3 = 0$ —— (2)

$-x_2 + 6x_3 = 0$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & -1 & 6 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-x_2 + 6x_3 = 0$

$6x_3 = x_2$

$x_3 = x_2/6$

$2x_1 - x_2 + 2x_3 = 0$

$2x_1 - x_2 + \frac{x_2}{3} = 2x_1 - \frac{2}{3}x_2 = 0$

$2x_1 = \frac{2}{3}x_2 \Rightarrow x_1 = -x_2$ $\quad X = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$

$\lambda = 10$

$$\begin{bmatrix} -6 & -2 & 4 \\ 0 & -12 & 12 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$R_3 \rightarrow 6R_3 - R_2$

$$\left[\begin{array}{ccc|c} -6 & 2 & 4 & 0 \\ 0 & -12 & 12 & 0 \\ 0 & 0 & 0 & 0 \\ & & & 0 \end{array}\right]$$

$-12x_2 = -12x_3$

$x_2 = x_3$

$-6x_1 - 2x_2 + 4x_3 = 0$

$-6x_1 + 2x_2 = 0$

$-6x_1 = -2x_2$

$x_2 = 3x_1$

$$\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 0 & 12 \\ -2 & 4-\lambda & 4 \\ -2 & 0 & 12-\lambda \end{vmatrix}$$

$= (-2-\lambda)\left[(4-\lambda)\cdot(12-\lambda)\right] + 12\left[2(4-\lambda)\right] = 0$

$4-\lambda\left[(-2-\lambda)(12-\lambda) + 24\right] = 0$

$4-\lambda = 0 \quad \lambda = 4$

$-24 + 2\lambda - 12\lambda + \lambda^2 + 24 = 0$

$\lambda^2 - 10\lambda = 0$

$\lambda(\lambda - 10) = 0$

$\lambda = 0, \ \lambda = 10$

At $\lambda = 4$

$$\begin{bmatrix} -6 & 0 & 12 \\ -2 & 0 & 4 \\ -2 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 12 & 0 \\ -2 & 0 & 4 & 0 \\ -2 & 0 & 8 & 0 \end{array}\right] \quad R_2 \rightarrow R_2 - R_3$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 12 & 0 \\ 0 & 0 & -4 & 0 \\ -2 & 0 & 8 & 0 \end{array}\right] \quad R_3 \rightarrow 3R_3 - R_1$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 12 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 12 & 0 \end{array}\right] \quad R_3 \rightarrow 3R_3 + R_4$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 12 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$-4x_3 = 0 \qquad x_3 = 0$

$-6x_1 + 12x_3 = 0$

$x_1 = 2x_3 = 0 \qquad X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$x_2 = 1$

At $\lambda = 0$

$-2x_1 + 12x_3 = 0$

$4x_2 = 8x_3$

$x_2 = 2x_3$

$X = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

At $\lambda = 10$

$-12x_1 + 12x_3 = 0$

$-2x_1 - 6x_2 + 4x_3 = 0$

$-2x_1 + 2x_3 = 0$

$$\begin{bmatrix} -12 & 0 & 12 & 0 \\ [illegible] \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

At $\lambda = 4$, we get λ as

$$\begin{bmatrix} 0 & -2 & 4 \\ 0 & -6 & 12 \\ 0 & -2 & 8 \end{bmatrix}$$

$-2x_2 + 4x_3$

$-6x_2 + 12x_3 = 0$

$-2x_2 + 8x_3 = 0$

$x_2 = 4x_3$

$-6x + 4x_3 + 12x_3 = 0$

$-12x_3 = 0$

$x_2 = 0$

Put $x_1 = 1$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let Y = eigen vector of λ.

at $\lambda = 0$

$$Y = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

$X = PY$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

At $\lambda = 4$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

At $\lambda = 10$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

Orthogonal Eigen Vectors

Let A be an $n \times n$ matrix and let eigen vector of A be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then corresponding orthonormal eigen vector is $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

eg: $\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{41} \\ 5/\sqrt{41} \end{bmatrix}$

Quadratic form

The quadratic form Q in the components $x_1, x_2 \dots x_n$ for a vector x is defined as $Q = X^T A X$ where A is the coefficient matrix.

for eg: Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A = \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$

$$Q = X^T A X.$$

Then corresponding quadratic form

$$Q = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 + 5x_2 & 5x_1 + 2x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 5x_1 x_2 + 5x_1 x_2 + 2x_2^2$$

$$Q = 3x_1^2 + 10x_1 x_2 + 2x_2^2$$

Consider the Quadratic form $Q = 3x_1^2 + 10x_1 x_2 + 2x_2^2$

$$A = \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$$

Let $\mathcal{X}$ be the matrix obtained by writing the orthogonal eigen vectors of A as column vectors.

$$\mathcal{X}^T x = Y$$

The quadratic form Q can be represented as

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 \quad \text{--- (1)}$$

where $\lambda_1, \lambda_2, \dots \lambda_n$ are eigen values of A.

$y_1, y_2 \dots y_n$ are components of y.

Here (1) is called principal axis form or cannonical form of the quadratic equation.

Qn: What kind of a conic section or pair of straight line is given by quadratic form:

Transform it to principal axis form and express $X^T = [x_1 \; x_2]$ in terms of new vector $Y^T = [y_1 \; y_2]$

1) $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

$$A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}$$

$$X^T x = y$$

Note: In quadratic form, the orthonormal eigen vectors are orthogonal.

ie $X^T = X^{-1}$

$$Q = \lambda_1 y_1^2 + \lambda_2 y^2$$

$$[A - \lambda I] = \begin{bmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{bmatrix} = 0$$

$$(7-\lambda)^2 - 9 = 0$$

$$40 + \lambda^2 - 14\lambda = 0$$

$\lambda = 4, 10$

$\therefore$ principal axis form is,

$$200 = 4y_1^2 + 10y_2^2$$

$$1 = \frac{y_1^2}{50} + \frac{y_2^2}{20} \rightarrow \text{It represents an ellipse}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

when $\lambda = 4$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$x_1 + x_2 = 0$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

when $\lambda = 10$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-x_1 + x_2 = 0$

$x_1 = x_2$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}} y_1 + 1/\sqrt{2}\, y_2$$

$$x_2 = 1/\sqrt{2}\, y_1 - 1/\sqrt{2}\, y_2$$

2) $x_1^2 - 12x_1x_2 + x_2^2 = 70$

$$A = \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix}$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$\begin{bmatrix} 1-\lambda & -6 \\ -6 & 1-\lambda \end{bmatrix} = 0$$

$(1-\lambda)^2 - 36 = 0$

$-35 + \lambda^2 - 2\lambda = 0 \qquad \lambda = 7, -5$

Principal axis form is

$$70 = 7y_1^2 - 5y_2^2$$

$$\frac{y_1^2}{10} - \frac{y_2^2}{14} = 1$$

Gam: a hyperbola.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\lambda = 7$

$$\begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$x_1 + x_2 = 0$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \quad X_1 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\lambda = -5$

$$\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1 - x_2 = 0$

$-x_1 + x_2 = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 = 0$

$x_1 = x_2$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_2 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$X = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$x_1 = 1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2 \Rightarrow x_2 = -1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2$

5) $3x^2 + 22x_1x_2 + 3x^2 = 0$

$$\begin{bmatrix} 3 & 11 \\ 11 & 3 \end{bmatrix} = A$$

$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$$\begin{bmatrix} 3-\lambda & 11 \\ 11 & 3-\lambda \end{bmatrix}$$

$(3-\lambda)^2 - 11^2 = 0$

$(3-\lambda) = \pm 11$

$\lambda = -8, 14$

when $\lambda = -8$

$$\begin{bmatrix} 11 & 11 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$x_1 + x_2 = 0$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\lambda = 14$

$$\begin{bmatrix} -11 & 11 \\ 11 & -11 \end{bmatrix}$$

$-x_1 + x_2 = 0$

$x_1 - x_2 = 0$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$

$$x_2 = -1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2$$

Equation for pair of straight lines

$[ax^2 + 2hxy + by^2 = 0]$

$-8y_1^2 + 14y_2^2 = 0$

Multiple Eigen Values

If an eigen value λ is repeating k times then we choose corresponding k eigen vectors linearly independent. Since the system is homogeneous, then the rank of row reduced matrix is r and if there are n variables then no: of linearly independent solution is (n-r) and remaining are zero vectors. No: K is called algebraic multiplicity of λ denoted by M_λ and (n-R) is called geometric multiplicity denoted by m_λ. Then $\Delta_\lambda = M_\lambda - m_\lambda$ is called defect of λ.

1) Find the value of (eigenvalue and eigen vector)

$$\begin{bmatrix} 6 & 5 & 2 \\ 2 & 0 & -8 \\ 5 & 4 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} (6-\lambda) & 5 & 2 \\ 2 & -\lambda & -8 \\ 5 & 4 & -\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(\lambda^2+32) - 5(-2\lambda+40) + 2(8+5\lambda) = 0$$

$$(6-\lambda)(\lambda^2+32) + 10\lambda - 200 + 16 + 10\lambda = 0$$

$$6\lambda^2 + 192 - 32\lambda - \lambda^3 + 20\lambda - 184.$$

$$-\lambda^3 + 6\lambda^2 - 12\lambda + 8 = 0$$

$$\lambda = 2, 2, 2.$$

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$4x_1 + 5x_2 + 2x_3 = 0$$

$$2x_1 - 2x_2 - 8x_3 = 0$$

$$5x_1 + 4x_2 - 2x_3 = 0$$

$$\left[\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 2 & -2 & -8 & 0 \\ 5 & 4 & -2 & 0 \end{array}\right]$$

$R_2 \rightarrow 2R_2 - R_1$

$$\begin{bmatrix} 4 & 5 & 2 & 0 \\ 0 & -9 & -18 & 0 \\ 5 & 4 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 5/4 R_1$

$$\begin{bmatrix} 4 & 5 & 2 & 0 \\ 0 & -9 & -18 & 0 \\ 0 & -9/4 & -18/4 & 0 \end{bmatrix}$$

$R_3 \rightarrow 4R_3 - R_2$

$$\left[\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 0 & -9 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$n = 3$

$r = 2$

$n - r = 1$

$-9x_2 - 18x_3 = 0$

$-9x_2 = 18x_3$

$x_2 = -2x_3$

$4x_1 - 10x_3 + 2x_3 = 0$

$4x_1 = 8x_3$

$x_1 = 2x_3$

$x_3 = 1$

$X_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Hint:

[$\lambda = 5, -3, -3$

$n - r = 2$

Put $x_1 = 0$]